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Bounds of the Normal Approximation for Linear Recursions with Two Effects

Mongkhon Tuntapthai*

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Abstract: Let X_0 be a non-constant random variable with finite variance. Given an integer $k \geq 2$, define a sequence $\{X_n\}_{n=1}^{\infty}$ of approximately linear recursions with small perturbations $\{\Delta_n\}_{n=0}^{\infty}$ by

$$X_{n+1} = \sum_{i=1}^{k} a_{n,i} X_{n,i} + \Delta_n \quad \text{for all } n \geq 0$$

where $X_{n,1}, \dots, X_{n,k}$ are independent copies of the X_n and $a_{n,1}, \dots, a_{n,k}$ are real numbers. In 2004, Goldstein obtained bounds on the Wasserstein distance between the standard normal distribution and the law of X_n which is in the form $C\gamma^n$ for some constants $C > 0$ and $0 < \gamma < 1$.

In this article, we extend the results to the case of two effects by studying a linear model $Z_n = X_n + Y_n$ for all $n \geq 0$, where $\{Y_n\}_{n=1}^{\infty}$ is a sequence of approximately linear recursions with an initial random variable Y_0 and perturbations $\{\Lambda_n\}_{n=0}^{\infty}$, i.e., for some $\ell \geq 2$,

$$Y_{n+1} = \sum_{j=1}^{\ell} b_{n,j} Y_{n,j} + \Lambda_n \quad \text{for all } n \geq 0$$

where Y_n and $Y_{n,1}, \dots, Y_{n,\ell}$ are independent and identically distributed random variables and $b_{n,1}, \dots, b_{n,\ell}$ are real numbers. Applying the zero bias transformation in the Stein's equation, we also obtain the bound for Z_n. Adding further

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conditions that the two models (X_n, Δ_n) and (Y_n, Λ_n) are independent and that the difference between variance of X_n and Y_n is smaller than the sum of variances of their perturbation parts, our result is the same as previous work.

Keywords: Hierarchical sequence, Stein's method, Zero bias

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1 Introduction and Main Theorem

Let Z be a standard normally distributed random variable and X_0 a non-constant random variable with finite variance. For a positive integer $k \geq 2$, we consider a sequence $\{X_n\}_{n=1}^{\infty}$ of approximately linear recursions with perturbations $\{\Delta_n\}_{n=0}^{\infty}$,

$$X_{n+1} = \sum_{i=1}^{k} a_{n,i} X_{n,i} + \Delta_n \qquad \text{for all } n \geq 0$$

where the X_n and $X_{n,1}, \dots, X_{n,k}$ are independent and identically distributed random variables and $a_{n,1}, \dots, a_{n,k}$ are real numbers. For all integers $n \geq 0$, we introduce some notation for the model (X_n, a_n, Δ_n),

$$\lambda_{a,n}^2 = \sum_{i=1}^{k} a_{n,i}^2, \quad \varphi_{a,n} = \sum_{i=1}^{k} \frac{|a_{n,i}|^3}{\lambda_{a,n}^3}, \quad \mathrm{Var}\,(X_n) = \sigma_{X,n}^2$$

and

$$\widetilde{X}_n = \frac{X_n - EX_n}{\sigma_{X,n}}.$$

Arising originally from statistical physics, the approximately linear recursions are special type of hierarchical strucutres and often applied to the conductivity of random mediums. A natural way in the classical probability theory is to study limit theorems for the distributions of these models. A strong law of large numbers for the hierarchical structure was obtained by [6, 4, 3]. The central limit theorem for recursions was first introduced by [7] and the bounds to normal approximation based on the Wasserstein distance were obtained by [2]. The following two conditions were used in the last work.

Condition 1.1. For each $i = 1, \dots, k$, the sequence $\{a_{n,i}\}_{n=0}^{\infty}$ converges to some real number a_i satisfying that at least two of the a_i's are nonzero. Set $\lambda_a^2 =$

$\sum_{i=1}^{k} a_i^2$. There exist $0 < \delta_{X,2} < \delta_{\Delta,2} < 1$ and positive constants $C_{X,2}$, $C_{\Delta,2}$ such that for all $n \geq 0$,

$$\operatorname{Var}(X_n) \geq C_{X,2}^2 \lambda_a^{2n} (1-\delta_{X,2})^{2n},$$

$$\operatorname{Var}(\Delta_n) \leq C_{\Delta,2}^2 \lambda_a^{2n} (1-\delta_{\Delta,2})^{2n}.$$

Condition 1.2. With $\delta_{X,2}$, $\delta_{\Delta,2}$ and λ_a as in the Condition 1.1, there exists $\delta_{X,4} \geq 0$ and $\delta_{\Delta,4} \geq 0$ such that

$$\phi_{X,\Delta,2} = \frac{(1-\delta_{\Delta,2})(1+\delta_{X,4})^3}{(1-\delta_{X,2})^4} < 1 \quad \text{and} \quad \phi_{X,\Delta,4} = \left(\frac{1-\delta_{\Delta,4}}{1-\delta_{X,2}}\right)^2 < 1$$

and positive constants $C_{X,4}$, $C_{\Delta,4}$ such that

$$E(X_n - EX_n)^4 \leq C_{X,4}^4 \lambda_a^{4n} (1+\delta_{X,4})^{4n},$$

$$E(\Delta_n - E\Delta_n)^4 \leq C_{\Delta,4}^4 \lambda_a^{4n} (1-\delta_{\Delta,4})^{4n}.$$

Recall that the Wasserstien distance or L^1-distance between two laws F and G is given by

$$\|F-G\|_1 = \int_{-\infty}^{\infty} |F(t)-G(t)|\, dt.$$

For any random variable X, the law or cumulative distribution function of X is denoted by $\mathcal{L}(X)$.

Theorem 1.3. [2] *Under Conditions 1.1 and 1.2, there exist constants $C > 0$ and $\gamma \in (0,1)$ such that*

$$\left\|\mathcal{L}(\widetilde{X}_n) - \mathcal{L}(Z)\right\|_1 \leq C\gamma^n.$$

In this article, we extend the bounds to the case of two effects. Let $\{Z_n\}_{n=0}^{\infty}$ be a sequence of linear model with two effects given by

$$Z_n = X_n + Y_n \quad \text{for all } n \geq 0$$

where Y_0 is a non-degenerated random and for some integer $\ell \geq 2$,

$$Y_{n+1} = \sum_{j=1}^{\ell} b_{n,j} Y_{n,j} + \Lambda_n \quad \text{for all } n \geq 0$$

where $b_{n,1},\ldots,b_{n,\ell}$ are real numbers, $Y_{n,1},\ldots,Y_{n,\ell}$ are independent copy of the Y_n and Λ_n is a small perturbation. Note that the perturbations Δ_n and Λ_n always depend on X_n and Y_n, respectively. From now on, we assume that random variables from two models of recursions (X_n,Δ_n) and (Y_n,Λ_n) are independent for all $n\geq 0$, and denote

$$\lambda_n^2=\sum_{i=1}^k a_{n,i}^2+\sum_{j=1}^{\ell} b_{n,j}^2,\quad \mathrm{Var}\,(Z_n)=\sigma_{X,n}^2+\sigma_{Y,n}^2=\sigma_n^2$$

and

$$\widetilde{Z}_n=\frac{Z_n-EZ_n}{\sigma_n}.$$

The bound for linear recursions with two effects is derived by adding further assumption that the difference between variances of two models (X_n,Δ_n), (Y_n,Λ_n), is smaller than variances of perturbations, the following is our main theorem.

Theorem 1.4. *With constants $\delta_{X,2}$, $\delta_{X,4}$, $\delta_{\Delta,2}$ and $\delta_{Y,2}$, $\delta_{Y,4}$, $\delta_{\Lambda,2}$ as in Condition 1.1 and 1.2 for the models (X_n,Δ_n) and (Y_n,Λ_n), suppose that*

$$\psi_{X,Y,\Lambda}=\frac{(1-\delta_{\Lambda,2})\,(1+\delta_{X,4})^3}{(1-\delta_{Y,2})\,(1-\delta_{X,2})^3}<1\quad \textit{and}\quad \psi_{Y,X,\Delta}=\frac{(1-\delta_{\Delta,2})\,(1+\delta_{Y,4})^3}{(1-\delta_{X,2})\,(1-\delta_{Y,2})^3}<1$$

and that

$$|\mathrm{Var}\,(X_n)-\mathrm{Var}\,(Y_n)|\leq\frac{\mathrm{Var}\,(\Delta_n)+\mathrm{Var}\,(\Lambda_n)}{\max\{\lambda_{a,n}^2,\lambda_{b,n}^2\}},$$

then there exist constants $C>0$ and $\gamma\in(0,1)$ such that

$$\left\|\mathcal{L}(\widetilde{Z}_n)-\mathcal{L}(Z)\right\|_1\leq C\gamma^n.$$

2 Auxiliary Results

Before proving the main theorem, we present some results for the models (X_n,Δ_n) and (Y_n,Λ_n). For all $n\geq 0$, let

$$r_{X,n}=\frac{\lambda_n\sigma_{X,n}}{\sigma_{n+1}},\quad r_{Y,n}=\frac{\lambda_n\sigma_{Y,n}}{\sigma_{n+1}}.$$

We begin with the bounds of $r_{X,n}$ and $r_{Y,n}$.

Lemma 2.1. *With constants* $\delta_{X,2}$, $\delta_{\Delta,2}$ *and* $\delta_{Y,2}$, $\delta_{\Lambda,2}$ *as in Condition 1.1 for the models* (X_n, Δ_n) *and* (Y_n, Λ_n), *and suppose that*

$$|\mathrm{Var}\,(X_n) - \mathrm{Var}\,(Y_n)| \leq \frac{\mathrm{Var}\,(\Delta_n) + \mathrm{Var}\,(\Lambda_n)}{\max\{\lambda_{a,n}^2, \lambda_{b,n}^2\}},$$

then for an integer $p \geq 1$, *there exists a positive constant* $C_{r,p}$ *such that*

$$\left| r_{X,n}^p - 1 \right| \leq C_{r,p} \left\{ \left(\frac{1-\delta_{\Delta,2}}{1-\delta_{X,2}} \right)^n + \left(\frac{1-\delta_{\Lambda,2}}{1-\delta_{Y,2}} \right)^n \right\}$$

and

$$\left| r_{Y,n}^p - 1 \right| \leq C_{r,p} \left\{ \left(\frac{1-\delta_{\Delta,2}}{1-\delta_{X,2}} \right)^n + \left(\frac{1-\delta_{\Lambda,2}}{1-\delta_{Y,2}} \right)^n \right\}.$$

Proof. Following the argument of [7, Lemma 6], we consider the variances of linear model of recursions

$$\begin{aligned} \sigma_{n+1}^2 &= \mathrm{Var}\,(Z_{n+1}) \\ &= \lambda_{a,n}^2 \mathrm{Var}\,(X_n) + \lambda_{b,n}^2 \mathrm{Var}\,(Y_n) + \mathrm{Var}\,(\Delta_n) + \mathrm{Var}\,(\Lambda_n) \\ &= \lambda_n^2 \sigma_{X,n}^2 + \lambda_{b,n}^2 \{\mathrm{Var}\,(Y_n) - \mathrm{Var}\,(X_n)\} + \mathrm{Var}\,(\Delta_n) + \mathrm{Var}\,(\Lambda_n), \end{aligned}$$

The triangle inequality yields

$$\begin{aligned} \sigma_{n+1} &\leq \lambda_n \sigma_{X,n} + \sqrt{\lambda_{b,n}^2 |\mathrm{Var}\,(Y_n) - \mathrm{Var}\,(X_n)|} + \sqrt{\mathrm{Var}\,(\Delta_n) + \mathrm{Var}\,(\Lambda_n)} \\ &\leq \lambda_n \sigma_{X,n} + 2\sqrt{\mathrm{Var}\,(\Delta_n) + \mathrm{Var}\,(\Lambda_n)}. \end{aligned}$$

Also, we note that

$$\begin{aligned} \lambda_{a,n}^2 \sigma_{X,n}^2 &= \sigma_{n+1}^2 - \lambda_{b,n}^2 \{\mathrm{Var}\,(Y_n) - \mathrm{Var}\,(X_n)\} - \mathrm{Var}\,(\Delta_n) - \mathrm{Var}\,(\Lambda_n) \\ &\leq \sigma_{n+1}^2 + \lambda_{b,n}^2 |\mathrm{Var}\,(Y_n) - \mathrm{Var}\,(X_n)| + \mathrm{Var}\,(\Delta_n) + \mathrm{Var}\,(\Lambda_n), \end{aligned}$$

which implies that

$$\begin{aligned} \lambda_n \sigma_{X,n} &\leq \sigma_{n+1} + \sqrt{\lambda_{b,n}^2 |\mathrm{Var}\,(Y_n) - \mathrm{Var}\,(X_n)|} + \sqrt{\mathrm{Var}\,(\Delta_n) + \mathrm{Var}\,(\Lambda_n)} \\ &\leq \lambda_n \sigma_{X,n} + 2\sqrt{\mathrm{Var}\,(\Delta_n) + \mathrm{Var}\,(\Lambda_n)}. \end{aligned}$$

Then there exists a constant $C_{r,1}$ such that

$$\begin{aligned}
|r_{X,n} - 1| &= \frac{|\lambda_n \sigma_{X,n} - \sigma_{n+1}|}{\sigma_{n+1}} \\
&\leq \frac{2\sqrt{\mathrm{Var}\,(\Delta_n) + \mathrm{Var}\,(\Lambda_n)}}{\sigma_{n+1}} \\
&\leq 2\sqrt{\frac{\mathrm{Var}\,(\Delta_n)}{\mathrm{Var}\,(X_{n+1})}} + 2\sqrt{\frac{\mathrm{Var}\,(\Lambda_n)}{\mathrm{Var}\,(Y_{n+1})}} \\
&\leq \frac{2C_{\Delta,2}\,(1-\delta_{\Delta,2})^n}{C_{X,2}\lambda_a\,(1-\delta_{X,2})^{n+1}} + \frac{2C_{\Lambda,2}\,(1-\delta_{\Lambda,2})^n}{C_{Y,2}\lambda_b\,(1-\delta_{Y,2})^{n+1}} \\
&\leq C_{r,1}\left\{\left(\frac{1-\delta_{\Delta,2}}{1-\delta_{X,2}}\right)^n + \left(\frac{1-\delta_{\Lambda,2}}{1-\delta_{Y,2}}\right)^n\right\}.
\end{aligned}$$

Now, since

$$|r^p - 1| = |(r-1+1)^p - 1| \leq \sum_{j=1}^{p} \binom{p}{j} |r-1|^j$$

and the assumption that $0 < \delta_{X,2} < \delta_{\Delta,2} < 1$ and $0 < \delta_{Y,2} < \delta_{\Lambda,2} < 1$, there are constants $C_{r,p}$ such that

$$\left|r_{X,n}^p - 1\right| \leq C_{r,p}\left\{\left(\frac{1-\delta_{\Delta,2}}{1-\delta_{X,2}}\right)^n + \left(\frac{1-\delta_{\Lambda,2}}{1-\delta_{Y,2}}\right)^n\right\}$$

and similarly, we can see that

$$\left|r_{Y,n}^p - 1\right| \leq C_{r,p}\left\{\left(\frac{1-\delta_{\Delta,2}}{1-\delta_{X,2}}\right)^n + \left(\frac{1-\delta_{\Lambda,2}}{1-\delta_{Y,2}}\right)^n\right\}$$

for all $p = 1, 2, 3, \ldots$. □

For all $n \geq 0$, let

$$U_n = U_{X,n} + U_{Y,n}$$

where

$$U_{X,n+1} = \sum_{i=1}^{k} \frac{a_{n,i}}{\lambda_n}\left(\frac{X_{n,i} - EX_{n,i}}{\sigma_{X,n}}\right) \quad \text{and} \quad U_{Y,n+1} = \sum_{j=1}^{\ell} \frac{b_{n,j}}{\lambda_n}\left(\frac{Y_{n,j} - EY_{n,j}}{\sigma_{Y,n}}\right).$$

Next, we follow the proof of [1, Lemma 4.1] to prepare an inequality for the Wasserstein distance between laws of U_n and its zero bias transformation.

Lemma 2.2. *For all integers $n \geq 1$ and the zero bias transformation U_n^*, $\widetilde{X}_n^*$, $\widetilde{Y}_n^*$ of the U_n, $\widetilde{X}_n$, $\widetilde{Y}_n$, respectively, we have*

$$\|\mathcal{L}(U_n) - \mathcal{L}(U_n^*)\|_1 \leq \left\|\mathcal{L}(\widetilde{X}_n) - \mathcal{L}(\widetilde{X}_n^*)\right\|_1 + \left\|\mathcal{L}(\widetilde{Y}_n) - \mathcal{L}(\widetilde{Y}_n^*)\right\|_1.$$

Proof. Set $m = k + \ell$. Let

$$\xi_i = \begin{cases} (X_{n,i} - EX_{n,i})/\sigma_{X,n} & \text{for } i = 1, \dots, k \\ (Y_{n,i-k} - EY_{n,i-k})/\sigma_{Y,n} & \text{for } i = k+1, \dots, m \end{cases}$$

and

$$\alpha_{n,i} = \begin{cases} a_{n,i} & \text{for } i = 1, \dots, k \\ b_{n,i-k} & \text{for } i = k+1, \dots, m. \end{cases}$$

Note that U_{n+1} is a sum of independent random variables and can be written as

$$U_{n+1} = \sum_{i=1}^{m} \frac{\alpha_{n,i}}{\lambda_n} \xi_i.$$

Let I be a random index independent of all other variables and satisfying that

$$P(I = i) = \frac{\alpha_{n,i}^2}{\lambda_n^2} \quad \text{for } i = 1, \dots, m.$$

By the result of [1, Lemma 2.8], the random variable

$$U_{n+1}^* = U_{n+1} - \frac{\alpha_{n,I}}{\lambda_n}(\xi_I^* - \xi_I)$$

has the U_{n+1}-zero biased distribution. By taking the dual form of the L^1-distance discussed in [5], we can see that

$$\left\|\mathcal{L}(U_{n+1}) - \mathcal{L}(U_{n+1}^*)\right\|_1 = \inf E|X - Y| \leq E\left|U_{n+1} - U_{n+1}^*\right|$$

where the infimum is taken over all coupling of X and Y having the joint distribution with $\mathcal{L}(U_{n+1})$ and its zero bias distribution.

Let $V_1, \dots, V_m$ be independent uniformly distributed random variables on $[0,1]$. For $i = 1, \dots, m$, let ξ_i^* be the zero bias transformation of ξ_i. Let F_ξ and F_{ξ^*} be the distribution functions of ξ and ξ^*, respectively. Set

$$(\xi_i, \xi_i^*) = \left(F_{\xi_i}^{-1}(V_i), F_{\xi_i^*}^{-1}(V_i)\right) \quad \text{for all } i = 1, \dots, m.$$

By the results of [5], we obtain that

$$E\,|\xi_i - \xi_i^*| = \begin{cases} \left\|\mathcal{L}(\widetilde{X}_n) - \mathcal{L}(\widetilde{X}_n^*)\right\|_1 & \text{for } i = 1, \ldots, k \\ \left\|\mathcal{L}(\widetilde{Y}_n) - \mathcal{L}(\widetilde{Y}_n^*)\right\|_1 & \text{for } i = k+1, \ldots, m. \end{cases}$$

Now, we obtain

$$\begin{aligned}
&\left\|\mathcal{L}(U_{n+1}) - \mathcal{L}(U_{n+1}^*)\right\|_1 \\
\leq\;& E\left|U_{n+1} - U_{n+1}^*\right| \\
=\;& E\sum_{i=1}^{m} \frac{|\alpha_{n,i}|}{\lambda_n}\,|\xi_i - \xi_i^*|\,\mathbf{1}\,(I = i) \\
=\;& \sum_{i=1}^{m} \frac{|\alpha_{n,i}|^3}{\lambda_n^3}\,E\,|\xi_i - \xi_i^*| \\
=\;& \sum_{i=1}^{k} \frac{|a_{n,i}|^3}{\lambda_n^3}\left\|\mathcal{L}(\widetilde{X}_n) - \mathcal{L}(\widetilde{X}_n^*)\right\|_1 + \sum_{j=1}^{\ell} \frac{|b_{n,j}|^3}{\lambda_n^3}\left\|\mathcal{L}(\widetilde{Y}_n) - \mathcal{L}(\widetilde{Y}_n^*)\right\|_1 \\
=\;& \frac{\lambda_{a,n}^3\,\varphi_{a,n}}{\lambda_n^3}\left\|\mathcal{L}(\widetilde{X}_n) - \mathcal{L}(\widetilde{X}_n^*)\right\|_1 + \frac{\lambda_{b,n}^3\,\varphi_{b,n}}{\lambda_n^3}\left\|\mathcal{L}(\widetilde{Y}_n) - \mathcal{L}(\widetilde{Y}_n^*)\right\|_1 \\
\leq\;& \left\|\mathcal{L}(\widetilde{X}_n) - \mathcal{L}(\widetilde{X}_n^*)\right\|_1 + \left\|\mathcal{L}(\widetilde{Y}_n) - \mathcal{L}(\widetilde{Y}_n^*)\right\|_1.
\end{aligned}$$

□

3 Proof of Main Theorem

Proof of Theorem 1.4. By the results of [1, Theorem 4.1], we can calculate the bound on L^1-distance by using the zero bias transformation as follows

$$\left\|\mathcal{L}(\widetilde{Z}_n) - \mathcal{L}(Z)\right\|_1 \;\leq\; 2\left\|\mathcal{L}(\widetilde{Z}_n) - \mathcal{L}(\widetilde{Z}_n^*)\right\|_1. \tag{3.1}$$

Moreover, we can use equivalent forms of the L^1-distance found in [5] and given by

$$\left\|\mathcal{L}(\widetilde{Z}_n) - \mathcal{L}(\widetilde{Z}_n^*)\right\|_1 = \sup_{h\in\mathfrak{Lip}}\left|Eh(\widetilde{Z}_n) - Eh(\widetilde{Z}_n^*)\right| = \sup_{f\in\mathfrak{F}_{\mathfrak{ac}}}\left|Ef'(\widetilde{Z}_n) - Ef'(\widetilde{Z}_n^*)\right|$$

where $\mathfrak{Lip} = \{h\colon \mathbb{R}\to\mathbb{R} : |h(x) - h(y)| \leq |x - y| \text{ for all } x, y \in \mathbb{R}\}$
and $\mathfrak{F}_{\mathfrak{ac}} = \{f\colon \mathbb{R}\to\mathbb{R} : f \text{ is absolutely continuous, } f(0) = f'(0) = 0,\ f' \in \mathfrak{Lip}\}$.

Now, we present some facts about the Stein's method for normal approximation. For each $f \in \mathcal{F}$, define $h\colon \mathbb{R} \to \mathbb{R}$ by

$$h(w) = f'(w) - wf(w).$$

By the characterization of normal distribution, $Eh(Z) = 0$. Also, we observe that

$$|h'(w)| \;=\; |f''(w) - wf'(w) - f(w)| \;\le\; 1 + w^2 + \frac{w^2}{2}$$

and hence

$$|h(w) - h(u)| = \left|\int_u^w h'(t)\,dt\right| \le |w-u| + \frac{1}{2}\left|w^3 - u^3\right|.$$

From the definition of zero bias transformation and that $\operatorname{Var}\left(\widetilde{Z}_{n+1}\right) = 1$, we have

$$\begin{aligned}
&\left|Ef'(\widetilde{Z}_{n+1}) - Ef'(\widetilde{Z}^*_{n+1})\right| \\
&= \left|Ef'(\widetilde{Z}_{n+1}) - E\widetilde{Z}_{n+1}f(\widetilde{Z}_{n+1})\right| \\
&= \left|Eh(\widetilde{Z}_{n+1})\right| \\
&\le \left|Eh(\widetilde{Z}_{n+1}) - Eh(U_{n+1})\right| + |Eh(U_{n+1})| \\
&\le E\left|\widetilde{Z}_{n+1} - U_{n+1}\right| + \frac{1}{2}E\left|\widetilde{Z}^3_{n+1} - U^3_{n+1}\right| + |Eh(U_{n+1})| \\
&= \beta_n + \left|Ef'(U_{n+1}) - Ef'(U^*_{n+1})\right| \\
&\le \beta_n + \left\|\mathcal{L}(U_{n+1}) - \mathcal{L}(U^*_{n+1})\right\|_1 \\
&\le \beta_n + \left\|\mathcal{L}(\widetilde{X}_n) - \mathcal{L}(\widetilde{X}^*_n)\right\|_1 + \left\|\mathcal{L}(\widetilde{Y}_n) - \mathcal{L}(\widetilde{Y}^*_n)\right\|_1 \qquad (3.2)
\end{aligned}$$

where we apply Lemma 2.2 in the last inequality and denote for all $n \ge 0$,

$$\beta_n = E\left|\widetilde{Z}_{n+1} - U_{n+1}\right| + \frac{1}{2}E\left|\widetilde{Z}^3_{n+1} - U^3_{n+1}\right|. \qquad (3.3)$$

By (3.1) and taking the supremum of (3.2) over $f \in \mathfrak{F}_{\mathfrak{ac}}$, we obtain

$$\begin{aligned}
\left\|\mathcal{L}(\widetilde{Z}_{n+1}) - \mathcal{L}(Z)\right\|_1 &\le 2\left\|\mathcal{L}(\widetilde{Z}_{n+1}) - \mathcal{L}(\widetilde{Z}^*_{n+1})\right\|_1 \\
&\le 2\beta_n + 2\left\|\mathcal{L}(\widetilde{X}_n) - \mathcal{L}(\widetilde{X}^*_n)\right\|_1 + 2\left\|\mathcal{L}(\widetilde{Y}_n) - \mathcal{L}(\widetilde{Y}^*_n)\right\|_1.
\end{aligned}$$

Applying the Condition 1.1 and 1.2 for the models (X_n, Δ_n) and (Y_n, Λ_n) in Theorem 1.3, there exist positive constants $C_{X,a,\Delta}$, $C_{Y,b\Lambda}$ and $\gamma_{X,a,\Delta} \in (0,1)$, $\gamma_{Y,b,\Lambda} \in (0,1)$ such that for all $n \ge 0$,

$$\left\|\mathcal{L}(\widetilde{X}_n) - \mathcal{L}(\widetilde{X}^*_n)\right\|_1 \le C_{X,a,\Delta}\left(\gamma_{X,a,\Delta}\right)^n$$

and

$$\left\|\mathcal{L}(\widetilde{Y}_n) - \mathcal{L}(\widetilde{Y}_n^*)\right\|_1 \leq C_{Y,b,\Lambda}\,(\gamma_{Y,b,\Lambda})^n\,.$$

We remain to show that $\beta_n \leq C_\beta \gamma_\beta^n$ for some $C_\beta > 0$ and $\gamma_\beta \in (0,1)$ and the proof is completed by choosing $C = C_{X,a,\Delta} + C_{Y,b,\Lambda} + C_\beta$ and $\gamma = \max\{\gamma_{X,a,\Delta}, \gamma_{Y,b,\Lambda}, \gamma_\beta\}$.

Recalling the definition of $r_{X,n}$, $r_{Y,n}$ and $U_{X,n}$, $U_{Y,n}$ in Lemma 2.1 and 2.2, respectively, the linear model of recursions can be written as

$$\begin{aligned}
\widetilde{Z}_{n+1} &= \frac{Z_{n+1} - EZ_{n+1}}{\sigma_{n+1}} \\
&= \frac{X_{n+1} - EX_{n+1}}{\sigma_{n+1}} + \frac{Y_{n+1} - EY_{n+1}}{\sigma_{n+1}} \\
&= \frac{\sigma_{X,n}}{\sigma_{n+1}} \left\{ \sum_{i=1}^{k} a_{n,i} \left(\frac{X_{n,i} - EX_{n,i}}{\sigma_{X,n}} \right) + \frac{\Delta_n - E\Delta_n}{\sigma_{X,n}} \right\} \\
&\quad + \frac{\sigma_{Y,n}}{\sigma_{n+1}} \left\{ \sum_{j=1}^{\ell} b_{n,j} \left(\frac{Y_{n,j} - EY_{n,j}}{\sigma_{Y,n}} \right) + \frac{\Lambda_n - E\Lambda_n}{\sigma_{Y,n}} \right\} \\
&= r_{X,n} U_{X,n+1} + r_{Y,n} U_{Y,n+1} + \Gamma_n
\end{aligned}$$

where $\Gamma_n = \Gamma_{X,\Delta,n} + \Gamma_{Y,\Lambda,n}$,

$$\Gamma_{X,\Delta,n} = \frac{\sigma_{X,n}}{\sigma_{n+1}} \left(\frac{\Delta_n - E\Delta_n}{\sigma_{X,n}} \right) \quad \text{and} \quad \Gamma_{Y,\Lambda,n} = \frac{\sigma_{Y,n}}{\sigma_{n+1}} \left(\frac{\Delta_n - E\Delta_n}{\sigma_{Y,n}} \right).$$

Using Conditions 1.1 and 1.2 for the models (X_n, Δ_n) and (Y_n, Λ_n), the result of [7, Lemma 6] gives that the limits

$$\lim_{n\to\infty} \frac{\sigma_{X,n}}{\lambda_{a,0} \ldots \lambda_{a,n-1}} \quad \text{and} \quad \lim_{n\to\infty} \frac{\sigma_{Y,n}}{\lambda_{b,0} \ldots \lambda_{b,n-1}}$$

exist in $(0,1)$, so we have

$$\lim_{n\to\infty} \frac{\sigma_{X,n+1}}{\sigma_{X,n}} = \lambda_a \quad \text{and} \quad \lim_{n\to\infty} \frac{\sigma_{Y,n+1}}{\sigma_{Y,n}} = \lambda_b.$$

Therefore, there exist positive constants $C_{\Gamma,X,\Delta,2}$ and $C_{\Gamma,Y,\Lambda,2}$ such that

$$\begin{aligned}
E\Gamma_{X,\Delta,n}^2 &\leq \left(\frac{\sigma_{X,n}}{\sigma_{X,n+1}} \right)^2 \frac{\mathrm{Var}\,(\Delta_n)}{\mathrm{Var}\,(X_n)} \leq C_{\Gamma,X,\Delta,2}^2 \left(\frac{1-\delta_{\Delta,2}}{1-\delta_{X,2}} \right)^{2n} \\
E\Gamma_{Y,\Lambda,n}^2 &\leq \left(\frac{\sigma_{Y,n}}{\sigma_{Y,n+1}} \right)^2 \frac{\mathrm{Var}\,(\Lambda_n)}{\mathrm{Var}\,(Y_n)} \leq C_{\Gamma,Y,\Lambda,2}^2 \left(\frac{1-\delta_{\Lambda,2}}{1-\delta_{Y,2}} \right)^{2n}.
\end{aligned}$$

Moreover, there exist positive constants $C_{\Gamma,X,\Delta,4}$ and $C_{\Gamma,Y,\Lambda,4}$ such that

$$E\Gamma^4_{X,\Delta,n} \le \left(\frac{\sigma_{X,n}}{\sigma_{X,n+1}}\right)^4 E\left(\frac{\Delta_n - E\Delta_n}{\sigma_{X,n}}\right)^4 \le C^4_{\Gamma,X,\Delta,4}\left(\frac{1-\delta_{\Delta,4}}{1-\delta_{X,2}}\right)^{4n}$$

$$E\Gamma^4_{Y,\Lambda,n} \le \left(\frac{\sigma_{Y,n}}{\sigma_{Y,n+1}}\right)^4 E\left(\frac{\Lambda_n - E\Lambda_n}{\sigma_{Y,n}}\right)^4 \le C^4_{\Gamma,Y,\Lambda,4}\left(\frac{1-\delta_{\Lambda,4}}{1-\delta_{Y,2}}\right)^{4n}.$$

By independence for $X_{n,i}$'s and $Y_{n,j}$'s, there exist positive constants $C_{U,X}$ and $C_{U,Y}$ such that

$$\begin{aligned}
EU^2_{X,n+1} &= \frac{\lambda^2_{a,n}}{\lambda^2_n} E\left(\frac{X_n - EX_n}{\sigma_{X,n}}\right)^2 \le 1 \\
EU^2_{Y,n+1} &= \frac{\lambda^2_{b,n}}{\lambda^2_n} E\left(\frac{Y_n - EY_n}{\sigma_{Y,n}}\right)^2 \le 1 \\
EU^4_{X,n+1} &\le 8\sum_{i=1}^{k} \frac{a^4_{n,i}}{\lambda^4_n} E\left(\frac{X_n - EX_n}{\sigma_{X,n}}\right)^4 \le C^4_{U,X}\left(\frac{1+\delta_{X,4}}{1-\delta_{X,2}}\right)^{4n} \\
EU^4_{Y,n+1} &\le 8\sum_{j=1}^{\ell} \frac{b^4_{n,j}}{\lambda^4_n} E\left(\frac{Y_n - EY_n}{\sigma_{Y,n}}\right)^4 \le C^4_{U,Y}\left(\frac{1+\delta_{Y,4}}{1-\delta_{Y,2}}\right)^{4n}.
\end{aligned}$$

From Lemma 2.1 and Condition 1.1 and 1.2, the following results will be often used for all $n \ge 0$ and $p = 1, 2, 3$,

$$\left| r^p_{X,n} - 1 \right| \le C_{r,p}\left(\phi^n_{X,\Delta,2} + \phi^n_{Y,\Lambda,2}\right) \tag{3.4}$$

$$\left| r^p_{Y,n} - 1 \right| \le C_{r,p}\left(\phi^n_{X,\Delta,2} + \phi^n_{Y,\Lambda,2}\right). \tag{3.5}$$

Now, considering the first term of β_n in (3.3),

$$\begin{aligned}
&E\left|\widetilde{Z}_{n+1} - U_{n+1}\right| \\
&= E\left|(r_{X,n}-1)U_{X,n+1} + (r_{Y,n}-1)U_{Y,n+1} + \Gamma_{X,\Delta,n} + \Gamma_{Y,\Lambda,n}\right| \\
&\le |r_{X,n}-1|\sqrt{EU^2_{X,n+1}} + |r_{Y,n}-1|\sqrt{EU^2_{Y,n+1}} + \sqrt{E\Gamma^2_{X,\Delta,n}} + \sqrt{E\Gamma^2_{Y,\Lambda,n}} \\
&\le 2C_{r,1}\left(\phi^n_{X,\Delta,2} + \phi^n_{Y,\Lambda,2}\right) + C_{\Gamma,X,\Delta,2}\left(\frac{1-\delta_{\Delta,2}}{1-\delta_{X,2}}\right)^n + C_{\Gamma,Y,\Lambda,2}\left(\frac{1-\delta_{\Lambda,2}}{1-\delta_{Y,2}}\right)^n \\
&\le C_0\left(\phi^n_{X,\Delta,2} + \phi^n_{Y,\Lambda,2}\right).
\end{aligned}$$

For the second term of β_n,

$$
\begin{aligned}
& E\left|\widetilde{Z}_{n+1}^3-U_{n+1}^3\right| \\
= \; & E\left|\left(r_{X,n}U_{X,n+1}+r_{Y,n}U_{Y,n+1}+\Gamma_n\right)^3-U_{n+1}^3\right| \\
= \; & E\left|\left(r_{X,n}U_{X,n+1}+r_{Y,n}U_{Y,n+1}\right)^3+3\left(r_{X,n}U_{X,n+1}+r_{Y,n}U_{Y,n+1}\right)^2\Gamma_n\right. \\
& \qquad \left.+3\left(r_{X,n}U_{X,n+1}+r_{Y,n}U_{Y,n+1}\right)\Gamma_n^2+\Gamma_n^3-U_{n+1}^3\right| \\
\leq \; & E\left|\left(r_{X,n}U_{X,n+1}+r_{Y,n}U_{Y,n+1}\right)^3-U_{n+1}^3\right|+3E\left|\left(r_{X,n}U_{X,n+1}+r_{Y,n}U_{Y,n+1}\right)^2\Gamma_n\right| \\
& +3E\left|\left(r_{X,n}U_{X,n+1}+r_{Y,n}U_{Y,n+1}\right)\Gamma_n^2\right|+E\left|\Gamma_n\right|^3 \\
:= \; & A_1+A_2+A_3+A_4.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
A_1 = \; & E\left|\left(r_{X,n}U_{X,n+1}+r_{Y,n}U_{Y,n+1}\right)^3-\left(U_{X,n+1}+U_{Y,n+1}\right)^3\right| \\
= \; & E\left|\left(r_{X,n}^3-1\right)U_{X,n+1}^3+3\left(r_{X,n}^2r_{Y,n}-1\right)U_{X,n+1}^2U_{Y,n+1}\right. \\
& \qquad \left.+3\left(r_{X,n}r_{Y,n}^2-1\right)U_{X,n+1}U_{Y,n+1}^2+\left(r_{Y,n}^3-1\right)U_{Y,n+1}^3\right| \\
\leq \; & E\left|\left(r_{X,n}^3-1\right)U_{X,n+1}^3\right|+3E\left|\left(r_{X,n}^2r_{Y,n}-r_{Y,n}+r_{Y,n}-1\right)U_{X,n+1}^2U_{Y,n+1}\right| \\
& +3E\left|\left(r_{X,n}r_{Y,n}^2-r_{X,n}+r_{X,n}-1\right)U_{X,n+1}U_{Y,n+1}^2\right|+E\left|\left(r_{Y,n}^3-1\right)U_{Y,n+1}^3\right| \\
\leq \; & E\left|\left(r_{X,n}^3-1\right)U_{X,n+1}^3\right|+E\left|\left(r_{Y,n}^3-1\right)U_{Y,n+1}^3\right| \\
& +3E\left|\left(r_{X,n}^2-1\right)r_{Y,n}U_{X,n+1}^2U_{Y,n+1}\right|+3E\left|\left(r_{Y,n}-1\right)U_{X,n+1}^2U_{Y,n+1}\right| \\
& +3E\left|\left(r_{Y,n}^2-1\right)r_{X,n}U_{X,n+1}U_{Y,n+1}^2\right|+3E\left|\left(r_{X,n}-1\right)U_{X,n+1}U_{Y,n+1}^2\right| \\
\leq \; & \left|r_{X,n}^3-1\right|\left(EU_{X,n}^4\right)^{3/4}+\left|r_{Y,n}^3-1\right|\left(EU_{Y,n}^4\right)^{3/4} \\
& +3\left|r_{X,n}^2-1\right|r_{Y,n}EU_{X,n}^2\sqrt{EU_{Y,n}^2}+3r_{X,n}\left|r_{Y,n}^2-1\right|\sqrt{EU_{X,n}^2}EU_{Y,n}^2 \\
& +3\left|r_{Y,n}-1\right|EU_{X,n}^2\sqrt{EU_{Y,n}^2}+3\left|r_{X,n}-1\right|\sqrt{EU_{X,n}^2}EU_{Y,n}^2 \\
\leq \; & 8^{3/4}C_{r,3}C_{X,4}^3\left\{\left(\frac{\left(1-\delta_{\Delta,2}\right)\left(1+\delta_{X,4}\right)^3}{\left(1-\delta_{X,2}\right)^4}\right)^n+\left(\frac{\left(1-\delta_{\Lambda,2}\right)\left(1+\delta_{X,4}\right)^3}{\left(1-\delta_{Y,2}\right)\left(1-\delta_{X,2}\right)^3}\right)^n\right\} \\
& +8^{3/4}C_{r,3}C_{Y,4}^3\left\{\left(\frac{\left(1-\delta_{\Delta,2}\right)\left(1+\delta_{Y,4}\right)^3}{\left(1-\delta_{X,2}\right)\left(1-\delta_{Y,2}\right)^3}\right)^n+\left(\frac{\left(1-\delta_{\Lambda,2}\right)\left(1+\delta_{Y,4}\right)^3}{\left(1-\delta_{Y,2}\right)^4}\right)^n\right\} \\
& +6C_{r,2}\left(1+2C_{r,1}\right)\left(\phi_{X,\Delta,2}^n+\phi_{Y,\Lambda,2}^n\right)+6C_{r,1}\left(\phi_{X,\Delta,2}^n+\phi_{Y,\Lambda,2}^n\right) \\
\leq \; & C_1\left(\phi_{X,\Delta,2}^n+\phi_{Y,\Lambda,2}^n+\psi_{X,Y,\Lambda}^n+\psi_{Y,X,\Delta}^n\right).
\end{aligned}
$$

As a special case of (3.4) and (3.5) when $p=1$, we can see that for all $n\geq 0$

$$r_{X,n}\leq 1+C_{r,1}\left(\phi^n_{X,\Delta,2}+\phi^n_{Y,\Lambda,2}\right)\leq 1+2C_{r,1}$$

$$r_{Y,n}\leq 1+C_{r,1}\left(\phi^n_{X,\Delta,2}+\phi^n_{Y,\Lambda,2}\right)\leq 1+2C_{r,1}.$$

So, we have that

$$\begin{aligned}
A_2 &= 3E\left|\left(r_{X,n}U_{X,n+1}+r_{Y,n}U_{Y,n+1}\right)^2\left(\Gamma_{X,\Delta,n}+\Gamma_{Y,\Lambda,n}\right)\right| \\
&\leq 6E\left|\left(r^2_{X,n}U^2_{X,n+1}+r^2_{Y,n}U^2_{Y,n+1}\right)\left(\Gamma_{X,\Delta,n}+\Gamma_{Y,\Lambda,n}\right)\right| \\
&\leq 6r^2_{X,n}\sqrt{EU^4_{X,n+1}E\Gamma^2_{X,\Delta,n}}+6r^2_{X,n}EU^2_{X,n+1}\sqrt{E\Gamma^2_{Y,\Lambda,n}} \\
&\quad +6r^2_{Y,n}EU^2_{Y,n+1}\sqrt{E\Gamma^2_{X,\Delta,n}}+6r^2_{Y,n}\sqrt{EU^4_{Y,n+1}E\Gamma^2_{Y,\Lambda,n}} \\
&\leq 12\sqrt{2}\left(1+C_{r,1}\right)^2C^2_{U,X}C_{\Gamma,X,\Delta,2}\left(\frac{\left(1-\delta_{\Delta,2}\right)\left(1+\delta_{X,4}\right)^3}{\left(1-\delta_{X,2}\right)^4}\right)^n \\
&\quad +6\left(1+C_{r,1}\right)^2C_{\Gamma,X,\Delta,2}\left(\frac{1-\delta_{\Delta,2}}{1-\delta_{X,2}}\right)^n+6\left(1+C_{r,1}\right)^2C_{\Gamma,Y,\Lambda,2}\left(\frac{1-\delta_{\Lambda,2}}{1-\delta_{Y,2}}\right)^n \\
&\quad +12\sqrt{2}\left(1+C_{r,1}\right)^2C^2_{U,Y}C_{\Gamma,Y,\Lambda,2}\left(\frac{\left(1-\delta_{\Lambda,2}\right)\left(1+\delta_{Y,4}\right)^3}{\left(1-\delta_{Y,2}\right)^4}\right)^n \\
&\leq C_2\left(\phi^n_{X,\Delta,2}+\phi^n_{Y,\Lambda,2}\right)
\end{aligned}$$

and that

$$\begin{aligned}
A_3 &= 3E\left|\left(r_{X,n}U_{X,n+1}+r_{Y,n}U_{Y,n+1}\right)\left(\Gamma_{X,\Delta,n}+\Gamma_{Y,\Lambda,n}\right)^2\right| \\
&\leq 6E\left|\left(r_{X,n}U_{X,n+1}+r_{Y,n}U_{Y,n+1}\right)\left(\Gamma^2_{X,\Delta,n}+\Gamma^2_{Y,\Lambda,n}\right)\right| \\
&\leq 6r_{X,n}\sqrt{EU^2_{X,n+1}E\Gamma^4_{X,\Delta,n}}+6r_{X,n}\sqrt{EU^2_{X,n+1}}E\Gamma^2_{Y,\Lambda,n} \\
&\quad +6r_{Y,n}\sqrt{EU^2_{Y,n+1}}E\Gamma^2_{X,\Delta,n}+6r_{Y,n}\sqrt{EU^2_{Y,n+1}E\Gamma^4_{Y,\Lambda,n}} \\
&\leq 6\left(1+C_{r,1}\right)\left(C^2_{\Gamma,X,\Delta,4}+C^2_{\Gamma,X,\Delta,2}\right)\left(\frac{1-\delta_{\Delta,2}}{1-\delta_{X,2}}\right)^{2n} \\
&\quad +6\left(1+C_{r,1}\right)\left(C^2_{\Gamma,Y,\Lambda,2}+C^2_{\Gamma,Y,\Lambda,4}\right)\left(\frac{1-\delta_{\Lambda,2}}{1-\delta_{Y,2}}\right)^{2n} \\
&\leq C_3\left(\phi^{2n}_{X,\Delta,2}+\phi^{2n}_{Y,\Lambda,2}\right).
\end{aligned}$$

Lastly,

$$\begin{aligned} A_4 &\leq \left(E\Gamma_n^4\right)^{3/4} \\ &\leq 8^{3/4}\left(E\Gamma_{X,\Delta,n}^4 + E\Gamma_{Y,\Lambda,n}^4\right)^{3/4} \\ &\leq 8^{3/4}\left\{C_{\Gamma,X,\Delta,4}^4\left(\frac{1-\delta_{\Delta,4}}{1-\delta_{X,2}}\right)^{4n} + C_{\Gamma,Y,\Lambda,4}^4\left(\frac{1-\delta_{\Lambda,4}}{1-\delta_{Y,2}}\right)^{4n}\right\}^{3/4} \\ &\leq C_4\left(\phi_{X,\Delta,4}^{3n/2} + \phi_{Y,\Lambda,4}^{3n/2}\right). \end{aligned}$$

Setting $\gamma_\beta = \max\left\{\phi_{X,\Delta,2}, \phi_{Y,\Lambda,2}, \phi_{X,\Delta,4}^{3/2}, \phi_{Y,\Lambda,4}^{3/2}, \psi_{X,Y,\Lambda}, \phi_{Y,X,\Delta}\right\} \in (0,1)$ and $C_\beta = 2C_0 + 4C_1 + 2C_2 + 2C_3 + 2C_4$, we obtain the claim for β_n. □

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Mongkhon Tuntapthai
Department of Mathematics, Faculty of Science,
Khon Kaen University
Email: mongkhon@kku.ac.th