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Technique in computing the characters of VOA modules using vector-valued functions for modular groups

Nakorn Junla

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Abstract: For a rational vertex operator algebra (VOA) V with a $\mathbb{Z}_{\geq 0}$-graded simple V-module M, there is a corresponding character

$$\text{ch } M = \text{Tr}_M q^{L_0^M - c/24} = \sum_n \dim M_n q^{n-c/24},$$

where M_n is the subspace of M on which L_0^M acts by multiplication by n, c is the central charge of V and $q = e^{2\pi i\tau}$.

In this paper, we apply the notion of vector-valued functions for a modular group from the work of Peter Bantay and Terry Gannon to compute the V-module character ch M. With the relation among simple V-modules M and their corresponding simple objects of modular tensor category (MTC) $\mathcal{C}$, we can use the central charge and conformal weights of the MTC $\mathcal{C}$ to compute the character ch M. This technique can be used to compute ch M up to central charge 24 by the restriction mentioned in P. Bantay paper.

Keywords: vector-valued functions, vertex operator algebra(VOA), Modular tensor category(MTC)

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1 Introduction

The character of a vertex operator algebra (VOA) is a generating function regarded as a formal q-expansion. This character contains some information of the VOA. We can find the character (formula) of a VOA directly from the structure of the VOA itself [7]. For VOAs with complicated structure, it is quite hard to find their characters (explicitly in q-expansion form) in this way. So we want to find another method to compute these characters. Fortunately with the work of Huang in [5], the category of certain types of VOA modules has a structure of a modular tensor category (MTC). So we can apply this fact to help us find the characters of the VOAs of this type.

One of the characteristics of a VOA is that it has a structure of a modular form [7]. With the idea from the work of P. Bantay in [1], we can develop a technique in computing the characters of the VOA modules via their counterparts, the MTC. Only certain types of the VOAs can be computed their characters in this way. We have to begin with a certain MTC and use some of its data in the computation. In this paper, we use the classification of the MTC as in Table 1 and have to restrict the central charge of the VOA to be at most 16.

This paper is divided into 2 parts. In Section 2, there are some basic facts of the VOAs, MTCs, and vector-valued modular forms. In Section 3, we develop the technique to compute the characters by applying the idea of vector-valued modular functions. The idea of the fundamental matrix in the work of P. Bantay is explained briefly (more detail in [1]). There are also some examples and results discussed in this section.

2 Preliminaries

2.1 Vertex Operator Algebras (VOAs)

For any $\mathbb{C}$-algebra $\mathbf{R}$, we denote by $\mathbf{R}[[z]]$ the space of $\mathbf{R}$-valued formal Taylor series in z. The space $\mathbf{R}((z))$ of $\mathbf{R}$-valued formal Laurent series in z is by definition the space of series $\sum_{n\in\mathbb{Z}} a_n z^n$, where $a_n \in \mathbf{R}$ for all n, and there exists $N \in \mathbb{Z}$ such that $a_n = 0, \forall n \leq N$.

Denoted by $\mathbb{C}((z))((w))$ the space $\mathbf{R}((w))$, where $\mathbf{R} = \mathbb{C}((z))$. In other words, this is the space of Laurent series in w whose coefficients are Laurent series in z.

Definition 2.1. A vertex algebra is a collection of data: (space of states) a vector space V; (vacuum vector) a vector $|0\rangle \in V$; (translation operator) a linear operator $T : V \to V$; (vertex operators) a linear operation $Y(\cdot, z) : V \to \mathrm{End}V[[z^{\pm 1}]]$ taking each $A \in V$ to a field acting on V,
$Y(A,z) = \sum_{n\in\mathbb{Z}} A_{(n)} z^{-n-1}$. These data are subject to the following axioms:

- (vacuum axiom) $Y(|0\rangle, z) = \mathrm{Id}_V$. Furthermore, for any $A \in V$ we have $Y(A,z)|0\rangle \in V[[z]]$. so that $Y(A,z)|0\rangle$ has a well-defined value at $z = 0$, and $Y(A,z)|0\rangle|_{z=0} = A$. In other words, $A_{(n)}|0\rangle = 0, n \geq 0$, and $A_{(-1)}|0\rangle = A$.
- (translation axiom) For any $A \in V$, $[T, Y(A,z)] = \partial_z Y(A,z)$ and $T|0\rangle = 0$.
- (locality axiom) All fields $Y(A,z)$ are local with respect to each other.

Let $K = \mathbb{C}((t))$ and $O = \mathbb{C}[[t]]$. Consider the Lie algebra $\mathrm{Der}\,K = \mathbb{C}((t))\partial_t$ of derivation of K. We define a **Virasoro vertex algebra** (see more [3]) as the central extension of $\mathrm{Der}\,K$: $0 \to \mathbb{C}C \to Vir \to \mathrm{DerK} \to 0$

It is known that this extension is universal. It has generators C, and
$L_n = -t^{n+1}\partial_t,\ n \in \mathbb{Z}$, satisfying the relations that C is central and

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3 - n}{12}\delta_{n,-m}C.$$

We will say that a module M over the Virasoro algebra has *central charge* $c \in \mathbb{C}$, if C acts on M by multiplication by c.

As the translation operator we take $T = L_{-1}$ and set $Y(L_{-2}v_C, z) \overset{def}{=} T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$.

$c = c(p,q) \overset{\mathrm{def}}{=} 1 - \dfrac{6(p-q)^2}{pq}$, $p, q > 1$, $(p,q) = 1$.

Definition 2.2. A $\mathbb{Z}_{\geq 0}$-graded vertex algebra V is called a **vertex operator algebra** (VOA), of central charge $c \in \mathbb{C}$, if we are given a non-zero conformal vector $\omega \in V_2$ such that the Fourier coefficients L_n^V of the corresponding vertex operator $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n^V z^{-n-2}$ satisfy the definition relations of the Virasoro algebra with central charge c, and in addition we have $L_{-1}^V = T$, $L_0^V|_{V_n} = n\mathrm{Id}$.

Definition 2.3. Let $(V, |0\rangle, T, Y)$ be a vertex algebra. A vector space M is called a $V-$*module* if it is equipped with an operation $Y_M : V \to \text{End}M[[z^{\pm 1}]]$ which assigns to each $A \in V$ a field $Y_M(A, z) = \sum_{n\in\mathbb{Z}} A^M_{(n)} z^{-n-1}$ on M subject to the following axioms: $Y_M(|0\rangle, z) = \text{Id}_M$; for all $A, B \in V$, $C \in M$ the three expressions $Y_M(A, z)Y_M(B, w)C \in M((Z))((w))$, $Y_M(B, w)Y_M(A, z)C \in M((w))((z))$, and $Y_M(Y(A, z-w)B, w)C \in M((w))((z))$ are the expressions, in their respective domains, of the same element of $M[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}]$.

If V is $\mathbb{Z}$-graded, then a V-module M is called *graded* if M is a $\mathbb{C}$-graded vector space and for $A \in V_m$ the field $Y_M(A, z)$ has conformal dimension m, i.e., the operator $A^M_{(n)}$ is homogeneous of degree $-n + m - 1$.

These axioms imply that V is a module over itself. A module M whose only submodules are 0 and itself is called *simple* or *irreducible*.

Definition 2.4. A VOA V is called ***rational*** if every $\mathbb{Z}_{\geq 0}$-graded V-module is completely reducible (i.e., isomorphic to a direct sum of simple V-modules).

This condition implies that V has *finitely many* inequivalent simple $\mathbb{Z}_{\geq 0}$-graded modules and the graded components of each simple $\mathbb{Z}_{\geq 0}$-graded V-module are finite dimensional.

If M is a simple $\mathbb{Z}_{\geq 0}$-graded V-module, then the Virasoro operator L_0^M on M is automatically semi-simple and hence defines a gradation on M. The above properties allow us to attach to a $\mathbb{Z}_{\geq 0}$-graded simple V-module M its *character*

$$\text{ch } M = \text{Tr}_M q^{L_0^M - c/24} = \sum_\alpha \text{dim} M_\alpha q^{\alpha - c/24},$$

where M_α is the subspace of M on which L_0^M acts by multiplication by α, c is the central charge of V, and $q = e^{2\pi i \tau}$.

Now let $C_2(V)$ be the subspace of V spanned by all elements of the form $A_{-2} \cdot B$ for all $A, B \in V$. Then a rational vertex algebra V is said to satisfy the C_2 *cofiniteness condition* if (1) $\dim V/C_2(V) < \infty$ and (2) every vector in V can be written as a linear combination of vectors of the form $L_{n_1} ... L_{n_k} A, n_i < 0$, where A satisfies $L_n A = 0$ for all $n > 0$.

Theorem 2.5 (cf.[11])**.** *Let V be a rational vertex algebra satisfying the C_2 cofiniteness condition, and let $\{M^1, ..., M^n\}$ be the set of all inequivalent simple $\mathbb{Z}$-graded V-modules (up to an isomorphism). Then the vector space spanned by* ch $M^i, i = 1, ..., n$, *is invariant under the action of $SL_2(\mathbb{Z})$.*

Definition 2.6. (cf. [4]) A VOA V is called unitary if V can be defined over the real numbers and the natural invariant symmetric form on it is positive definite.

2.2 Modular Tensor Categories (MTCs)

Definition 2.7. [MTCs] (see full detail in [10]) A modular category is a pair consisting of a ribbon Ab-category $\mathcal{C}$ and a finite family $\{V_i\}_{i\in I}$ of simple objects of $\mathcal{C}$ satisfying the following four axioms ; (1) There exists $0 \in I$ such that $V_0 = \mathbf{1}$; (2) For any $i \in I$, there exists $i^* \in I$ such that the object V_{i^*} is isomorphic to $(V_i)^*$; (3) All objects of $\mathcal{C}$ are dominated by the family $\{V_i\}_{i\in I}$ and (4) The square matrix $S = [S_{i,j}]_{i,j\in I}$ is invertible over K.

Remark 2.1. (see [8]) If $\{V_0 = \mathbf{1}, V_1, ..., V_{n-1}\}$ is the set of representatives of the simple objects in $\mathcal{C}$, the *rank* of $\mathcal{C}$ is n. We have $V_i \otimes V_j \cong \sum_k N_{i,j}^k V_k$ for some $N_{i,j}^k \in \mathbb{N}$.

The first column (and row) of the matrix S consists of the categorical dimensions of the simple objects, i.e., $S_{i,0} = \dim(V_i)$. We denote these dimensions by d_i. Since the twist $\theta_V \in \text{End}(V)$ for any object V, θ_V is a scalar map. We denote this scalar by θ_i.

If we set $T = (\delta_{i,j}\theta_i)_{ij}$ then the map: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \to S$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to T$ defines a projective representation of the *modular group* $SL_2(\mathbb{Z})$.

2.3 Vector Valued Modular Forms

Let $\rho : \Gamma \to GL_d(\mathbb{C})$ denote a d-dimensional representation of $\Gamma = SL_2(\mathbb{Z})$, $k \in \mathbb{R}$ an arbitrary real number. A function $F(\tau) = \begin{pmatrix} f_1(\tau) \\ \vdots \\ f_d(\tau) \end{pmatrix}$, where $\tau \in \mathbb{H}$ from the complex upper half-plane $\mathbb{H}$ to $\mathbb{C}^d$ is a ***vector-valued modular form*** of weight k if the following conditions are satisfied : (1) For all $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have $F(\tau) \mid_k V(\tau) = \rho(V)F(\tau)$ and (2) Each component function $F_j(\tau)$ has a convergent q-expansion meromorphic at infinity $F_j(\tau) = \sum_{n>h_j} a_n(j)q^{\frac{n}{N_j}}$ with N_j a positive integer, h_j an integer (maybe negative) and $q = \exp(2\pi i\tau)$.

The operator $|_k V$ is defined by

$$F \mid_k V(\tau) = F \mid_k^v V(\tau) = v(V^{-1})(c\tau + d)^{-k}F(V_\tau)$$

with a multiplier system v with respect to Γ. (see more in [6]).

3 Main Result

In this section, we will explain how to compute the characters of the VOA modules corresponding to small modular tensor categories(MTCs) using the method of vector-valued function for modular group which has been discussed in [1]. By small modular tensor categories we mean the MTCs of rank less than or equal to 4. Note that the VOAs that we consider here have to satisfy the conditions in Theorem 3.1 below and they are assumed to be unitary.

Theorem 3.1. *(cf. theorem 4.6 in [5]) Let V be a simple vertex operator algebra. Assume that : (1) $V_n = 0$ for $n < 0$, $V_0 = \mathbb{C}\mathbf{1}$ and V' is isomorphic to V as a V-module; (2) every $\mathbb{N}$-graded weak V-module is completely reducible; (3) V satisfy the C_2 cofiniteness condition.*

Then the category of V-modules has a natural structure of modular tensor category.

Recall that the VOA has a finite number of simple modules $V = M^1$, $M^2, \ldots M^r$. Each has a q-graded character

$$\text{ch } M^j = \text{Tr}_{M^j} q^{L_0^{M^j} - c/24} = \sum_n \dim M_n^j q^{n-c/24} = q^{h_j - c/24} \sum_n \dim M_{n+h_j}^j q^n, \tag{3.1}$$

where M_n^j is the subspace of M^j on which $L_0^{M^j}$ acts by multiplication by n, c is the central charge of V, h_j is the *conformal weight* of M^j, and $q = e^{2\pi i\tau}$. And these VOA modules have the structure of the modular tensor categories. So we can compute the characters the VOA modules using their associated MTCs.

Proposition 3.1 in [2] states that for each state $u \in V$ which is homogeneous of weight k with respect to the operator L_0, the r-tuple $Z(u,\tau) = (Z_{M^1}(u,\tau), \ldots, Z_{M^r}(u,\tau))$ is a *vector-valued modular form* of weight k with respect to the representation ρ.

Note that $Z_{M^j}(u,\tau) = \text{Tr}_{M^j} o(u) q^{L_0^{M^j} - c/24} = q^{h_j - c/24} \sum_n \dim M_{n+h_j}^j o(u) q^n$, where $o(u)$ is the zero mode of the homogeneous components of u (see detail in [2]).

Let V be a rational VOA satisfies the conditions in Theorem 3.1. Denote that $\mathcal{C}(V)$ be the MTC corresponding to the VOA V.

The family $\{\text{ch } M^i\}_{i=1,..n}$ is a vector valued modular function of a representation ρ of $SL_2(\mathbb{Z})$ determined by $\mathcal{C}(V)$. Note that h_i (mod 1) is given by $\mathcal{C}(V)$ and for a unitary VOA V, $h_i \geq 0$ and $c \geq 0$.

Let $M(\rho, c)$ be the space consisting of vector valued modular forms for the representation ρ with pole orders at most $c/24$ at infinity. Then ch M^i is an element of $M(\rho, c)$ and $M(\rho, c)$ depends only on the genus of V. Our objective is to use this fact and the work in [1] to compute the characters ch M^i, $i = 1, ..n$ of the VOA V.

3.1 Small MTCs

The following table consists of the list of MTCs of rank 1, 2, 3, and 4 which we call small MTCs. We use the classification of the MTCs from [9].

Table 1: The small MTCs

No.	$\mathcal{C}$	n	c (mod 8)	h_i
1	t_m	1	0	0
2	qs_2	2	1	0, 1/4
3	$\overline{qs_2}$	2	7	0, 3/4
4	$Lee - Yang$	2	14/5	0, 2/5
5	$\overline{Lee - Yang}$	2	26/5	0, 3/5
6	qs_3	3	2	0, 1/3, 1/3
7	$\overline{qs_3}$	3	6	0, 2/3, 2/3
8	$Ising1$	3	1/2	0, 1/2, 1/16
9	$\overline{Ising1}$	3	15/2	0, 1/2, 15/16
10	$Ising2$	3	3/2	0, 1/2, 3/16
11	$\overline{Ising2}$	3	13/2	0, 1/2, 13/16
12	$Ising3$	3	5/2	0, 1/2, 5/16
13	$\overline{Ising3}$	3	11/2	0, 1/2, 11/16
14	$Ising4$	3	7/2	0, 1/2, 7/16
15	$\overline{Ising4}$	3	9/2	0, 1/2, 9/16
16	$3fieldsx$	3	8/7	0, 2/7, 6/7
17	$\overline{3fieldsx}$	3	48/7	0, 5/7, 1/7
18	qs_4	4	1	0, 1/8, 1/8, 1/2
19	$\overline{qs_4}$	4	7	0, 7/8, 7/8, 1/2
20	qn_4	4	5	0, 1/2, 5/8, 5/8
21	$\overline{qn_4}$	4	3	0, 3/8, 3/8, 1/2
22	qu_2	4	8	0, 0, 0, 1/2
23	qv_2	4	4	0, 1/2, 1/2, 1/2
24	$qs_2 \otimes qs_2$	4	2	0, 1/4, 1/4, 1/2
25	$\overline{qs_2} \otimes \overline{qs_2}$	4	6	0, 3/4, 3/4, 1/2
26	$qs_2 \otimes \overline{qs_2}$	4	8	0, 3/4, 1/4, 1
27	$qs_2 \otimes LY$	4	19/5	0, 2/5, 1/4, 13/20

Continued on next page

Table 1 – *Continued from previous page*

No.	$\mathcal{C}$	n	c (mod 8)	h_i
28	$\overline{qs_2} \otimes LY$	4	49/5	0, 2/5, 3/4, 3/20
29	$qs_2 \otimes \overline{LY}$	4	31/5	0, 3/5, 1/4, 17/20
30	$\overline{qs_2} \otimes \overline{LY}$	4	61/5	0, 3/5, 3/4, 7/20
31	$LY \otimes LY$	4	28/5	0, 2/5, 2/5, 4/5
32	$LY \otimes \overline{LY}$	4	8	0, 3/5, 2/5, 1
33	$\overline{LY} \otimes \overline{LY}$	4	52/5	0, 3/5, 3/5, 1/5
34	$4fieldsx$	4	10/3	0, 2/3, 2/9, 1/3
35	$\overline{4fieldsx}$	4	14/3	0, 1/3, 7/9, 2/3

From the table, column 2 ($\mathcal{C}$) consists of the names of the MTCs which we follow the notation from the database [12]. The rank and the central charge (mod 8) of each MTC is shown in column 3 (n) and 4 (c) respectively. And the last column (h_i) consists of the conformal weights of each MTC.

Recall that there exists a representation $\rho : SL_2(\mathbb{Z}) \longleftrightarrow GL_n(\mathbb{C})$ of the modular group $SL_2(\mathbb{Z})$ sending its generating elements, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, to the matrices S and T of a MTC.

The representation ρ could be either irreducible or reducible. For a reducible representation ρ, we have to decompose it into a direct sum $\rho = \rho_1 \oplus \cdots \oplus \rho_s$ of its irreducible components ρ_i for our method.

We use Magma to decompose the representation ρ into a direct sum of its irreducible components, and we also get the corresponding canonical basis vectors for each irreducible representation ρ_i. Note that the idea of the decomposition is also mentioned in the appendix in [1].

3.2 The Fundamental Matrix

We give a brief detail of the fundamental matrix of the representation as in [1] using the same definitions, theorems, symbols as in [1].

Consider a matrix representation $\rho : \Gamma \to GL_d(\mathbb{C})$ whose kernel contains $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and for which $T = \rho\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is a diagonal matrix of finite order. We associate to ρ the set $\mathcal{M}(\rho)$ of all those maps $\mathbb{X} : \mathbb{H} \to \mathbb{C}^d$ which are holomorphic in the upper half plane $\mathbb{H}$, transform according to ρ, that is

$$\mathbb{X}\left(\frac{a\tau+b}{c\tau+d}\right) = \rho\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\mathbb{X}(\tau)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$, and have only finite order poles at the cusps. So $\mathbb{X}$ is a vector-valued modular form. There exists a diagonal matrix Λ (the ***exponent matrix***) such that $\rho\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = \exp(2\pi i\Lambda)$, the diagonal elements of Λ being rational numbers.

The space $\mathcal{M}(\rho)$ is an infinite-dimensional linear space over $\mathbb{C}$, a basis being provided by the maps $\mathbb{X}^{(\xi,n)} \in \mathcal{M}(\rho)$ which have a pole of order $n > 0$ at the ξth position. We call these $\mathbb{X}^{(\xi,n)}$ the *canonical basis vectors.*

Let $J(\tau) = q^{-1} + \sum_{n=1}^{\infty} c(n)q^n = q^{-1} + 196884q + ...$ denote the Hauptmodul of $SL_2(\mathbb{Z})$. Multiplication by J takes the space $\mathcal{M}(\rho)$ to itself, in other words $\mathcal{M}(\rho)$ is a $\mathbb{C}[J]$-module of finite rank and the canonical basis vectors satisfy the *recursion relations*

$$\mathbb{X}^{(\xi;m+1)} = J(\tau)\mathbb{X}^{(\xi,m)} - \sum_{n=1}^{m-1} c(n)\mathbb{X}^{(\xi,m-n)} - \sum_{\eta} \mathbf{x}_{\eta}^{(\xi;m)}\mathbb{X}^{(\eta;1)}, \tag{3.2}$$

where

$$\mathbf{x}_{\eta}^{(\xi;m)} = \mathbb{X}^{(\xi;m)}[0]_{\eta} = \lim_{q\to 0}\left([q^{-\Lambda}\mathbb{X}^{(\xi;m)}(q)]_{\eta} - q^{-m}\delta_{\xi\eta}\right) \tag{3.3}$$

denotes the "constant part" of $\mathbb{X}^{(\xi;m)}$. These recursion relations allow us to express each canonical basis vector $\mathbb{X}^{(\xi;m)}$ in terms of the $\mathbb{X}^{(\xi,1)}$s.

There is a second set of relations, "the *differential relations*", between the canonical basis vectors. They follow from the fact that the differential operator

$$\nabla = \frac{\mathcal{E}(\tau)}{2\pi i}\frac{d}{d\tau} \tag{3.4}$$

maps $\mathcal{M}(\rho)$ to itself, where

$$\mathcal{E}(\tau) = \frac{E_{10}(\tau)}{\Delta(\tau)} = \sum_{n=-1}^{\infty} \mathcal{E}_n q^n = q^{-1} - 240 - 141444q - ... \tag{3.5}$$

is the quotient of the Eisenstein series of weight 10 by the discriminant form $\Delta(\tau) = q\Pi_{n=1}^{\infty}(1-q^n)^{24}$ of weight 12. The action of ∇ on the canonical basis vectors gives the differential relations

$$\nabla\mathbb{X}^{(\xi;m)} = (\Lambda_{\xi\xi} - m)\sum_{n=-1}^{m-1} \mathcal{E}_n\mathbb{X}^{(\xi;m-n)} + \sum_{\eta} \Lambda_{\eta\eta}\mathbf{x}_{\eta}^{(\xi;m)}\mathbb{X}^{(\eta;1)}. \tag{3.6}$$

The compatibility of the recursion and differential relations requires that

$$\nabla \mathbb{X}^{(\xi;1)} = (J-240)(\Lambda_{\xi\xi}-1)\mathbb{X}^{(\xi;1)} + \sum_{\eta}(1+\Lambda_{\eta\eta}-\Lambda_{\xi\xi})\mathbf{x}_{\eta}^{(\xi;1)}\mathbb{X}^{(\eta;1)}, \qquad (3.7)$$

which is a first-order ordinary differential equation - the *compatibility equation* - for the $\mathbb{X}^{(\xi;1)}$s.

From equation (3.7), we define the ***fundamental matrix*** as follow

$$\Xi(\tau)_{\xi\eta} = [\mathbb{X}^{(\eta;1)}(\tau)]_{\xi}, \qquad (3.8)$$

whose columns span over $\mathbb{C}[J]$ the module $\mathcal{M}(\rho)$. Then equation (3.7) takes the form

$$\frac{1}{2\pi i}\frac{d\Xi(\tau)}{d\tau} = \Xi(\tau)\mathfrak{D}(\tau), \qquad (3.9)$$

where

$$\mathfrak{D}(\tau) = \frac{1}{\mathcal{E}(\tau)}\{(J(\tau)-240)(\Lambda-1)+\mathbf{x}+[\Lambda,\mathbf{x}]\}, \qquad (3.10)$$

$\mathbf{x}_{\xi\eta} = \mathbf{x}_{\xi}^{(\eta;1)}$ is the ***characteristic matrix*** and $[\mathbf{x},\Lambda] = \mathbf{x}\Lambda - \Lambda\mathbf{x}$.

Taking the boundary condition

$$q^{1-\Lambda_{\xi\xi}}\Xi(q)_{\xi\eta} = \delta_{\xi\eta} + O(q) \text{ as } q \to 0, \qquad (3.11)$$

one can solve equation (3.9), provided one knows the exponent matrix Λ and the characteristic matrix $\mathbf{x}$, determining then from equation (3.2) the canonical basis vectors $\mathbb{X}^{(\xi;m)}$.

Note that the exponent matrix has to satisfy the following condition:

$$\mathrm{Tr}(\Lambda) = \frac{5d}{12} + \frac{1}{4}\mathrm{Tr}(S) + \frac{2}{3\sqrt{3}}\mathrm{Re}(e^{-\pi i/6}\mathrm{Tr}(U)) \qquad (3.12)$$

where d is the dimension of ρ and we use the notations

$$S = \rho\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \text{ and } U = \rho\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$$

. The structure of the $\mathbb{C}[J]$-module $\mathcal{M}(\rho)$ is completely determined by the fundamental matrix $\Xi(\tau)$ (cf. [1]), once an exponent matrix Λ has been chosen. The fundamental matrix is itself completely determined by the pair $(\Lambda,\mathbf{x})$ of exponent and characteristic matrices, namely as the solution of the compatibility equation (3.9) satisfying the boundary condition equation (3.11). We consider the pair $(\Lambda,\mathbf{x})$ as the basis data characterizing the representation ρ.

Remark 3.1. The fundamental matrix Ξ allows us to determine the space $M(\rho, c) \subset \mathcal{M}(\rho)$ for $c < 24$.

3.3 Method of finding the Fundamental Matrix

Consider the function $\mathfrak{j}(\tau) = \dfrac{984 - J(\tau)}{1728}$, which maps the upper half-plane $\mathbb{H}$ onto the complex plane $\mathbb{C}$. It is modular invariant and satisfies the differential equation $\nabla \mathfrak{j} = 1728\mathfrak{j}(\mathfrak{j} - 1)$.

Let us consider the fundamental matrix as a function of $\mathfrak{j}$. Then, by applying the chain rule and last equation above, one gets the following form of the compatibility equation; $\dfrac{d\Xi(\mathfrak{j})}{d\mathfrak{j}} = \Xi\left(\dfrac{\mathcal{A}}{2\mathfrak{j}} + \dfrac{\mathcal{B}}{3(\mathfrak{j}-1)}\right)$, with

$$\mathcal{A} = \frac{31}{36}(1 - \Lambda) - \frac{1}{864}(\mathbf{x} + [\Lambda, \mathbf{x}]), \tag{3.13}$$

$$\mathcal{B} = \frac{41}{24}(1 - \Lambda) + \frac{1}{576}(\mathbf{x} + [\Lambda, \mathbf{x}]). \tag{3.14}$$

There are some restrictions for the matrices $\mathcal{A}$ and $\mathcal{B}$ called **Spectral condition** (see full detail in [1]).

With this condition, we get $\mathcal{A}(\mathcal{A} - 1) = \mathcal{B}(\mathcal{B} - 1)(\mathcal{B} - 2) = 0$.

Of the four matrices $\Lambda, \mathbf{x}, \mathcal{A}$ and $\mathcal{B}$, any two determine the other two, e.g., equations (3.13) and (3.14) imply that $\mathcal{B} = 3(1 - \Lambda - \mathcal{A}/2)$. Inserting this expression into equation (3.9), one gets the following system of algebraic equations:

$$\begin{aligned} &\mathcal{A}^2 = \mathcal{A}, \\ &\mathcal{A}\Lambda\mathcal{A} = -\frac{17}{18}\mathcal{A} - 2(\mathcal{A}\Lambda^2 + \Lambda\mathcal{A}\Lambda + \Lambda^2\mathcal{A}) + 3(\mathcal{A}\Lambda + \Lambda\mathcal{A}) - 4\Lambda^3 + 8\Lambda^2 - \frac{44}{9}\Lambda + \frac{8}{9} \end{aligned} \tag{3.15}$$

That is, for a given exponent matrix Λ, the matrix $\mathcal{A}$ has to satisfy equations (3.15). Once a solution to equations (3.15) is known, the corresponding characteristic matrix $\mathbf{x}$ may be determined from equation (3.13).

To find the fundamental matrix $\Xi(\tau)$, we do the following:

- Begin from a given exponent matrix Λ and then solve the equations (3.15) to get the matrix $\mathcal{A}$.
- Use the matrix $\mathcal{A}$ to find the characteristic matrix $\mathbf{x}$ by solving equation (3.13).
- Use the exponent matrix Λ and the characteristic matrix $\mathbf{x}$ to get the fundamental matrix $\Xi(\tau)$ by solving equation (3.9).

Recall that the matrices S and T of any MTC correspond to some representation of the modular group $SL_2(\mathbb{Z})$. In particular, they are the images of the representation ρ of the generators of the modular group.

For a given MTC $\mathcal{C}$ of rank n with the central charge c and conformal weights $h_1, h_2, ..h_n$. We define $\lambda_i = h_i - \dfrac{c}{24}$ and set

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & \ddots & & \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}.$$

Then Λ is the exponent matrix corresponding to the MTC $\mathcal{C}$.

Note that we may have to modify some of the λ_is to be $\lambda_i \bmod 1$ in order that Λ satisfies equation (3.12).

Follow the method above, one gets the corresponding fundamental matrix Ξ of the representation of the MTC $\mathcal{C}$. In this paper, we explore only the small MTCs up to $n = 4$.

3.4 Results

Now we will give some examples of how to compute the fundamental matrices and the corresponding characters from a given MTC.

Example 3.2. The characters of the VOA modules with central charge 1 corresponding to the MTC qs_2 of rank 2.

The MTC qs_2 is of rank 2 with central charge 1 and conformal weights 0, 1/4. We have $\lambda_1 = 0 - 1/24 = -1/24$ and $\lambda_2 = 1/4 - 1/24 = 5/24$ and the exponent matrix is $\Lambda = \begin{pmatrix} \frac{23}{24} & 0 \\ 0 & \frac{5}{24} \end{pmatrix}$.

Note that Λ has to satisfy some certain conditions (see [1]) so we have to modify λ_1 to λ_1 mod 1. Next we solve the equations (3.15) to get the matrix $\mathcal{A}$.

We get $\mathcal{A} = \begin{pmatrix} \frac{7}{216} & f(1,2) \\ \frac{1463}{46656 f(1,2)} & \frac{209}{216} \end{pmatrix}$.

Note that $f(1,2)$ is a parameter since there are infinitely many possible solutions for the equations (3.15).

Next we solve the equation (3.13) to get the characteristic matrix with the parameter; $\mathbf{x} = \begin{pmatrix} 3 & -\frac{3456}{7} f(1,2) \\ -\frac{2926}{27 f(1,2)} & -247 \end{pmatrix}$.

Finally, we solve the equation (3.9) and get the fundamental matrix with the parameter as follow

$$\Xi = q^{\Lambda} \begin{pmatrix} q^{-1}+3+4q+7q^2+\cdots & -\frac{3456}{7} f(1,2) - \frac{2464128}{77} f(1,2) q - \cdots \\ -\frac{2926}{27 f(1,2)} - \frac{2926q}{27 f(1,2)} - \cdots & q^{-1}-247-86241q-\cdots \end{pmatrix}.$$

To find the value of the parameter $f(1,2)$, we compare the first column of the fundamental matrix with the known characters of the Wess-Zumino-Novikov-Witten (WZW) model of level 1 based on the corresponding Lie algebra (see [1]). In this case, qs_2 corresponds to the affine Kac-Moody Lie algebra $A_{1,1}$ (the WZW model A_1 level 1). By comparing the first column of Ξ with the corresponding characters of $A_{1,1}$, we get the following result.

The characteristic matrix is $\mathbf{x} = \begin{pmatrix} 3 & 26752 \\ 2 & -247 \end{pmatrix}$

and the fundamental matrix is

$$\Xi = q^{\Lambda} \begin{pmatrix} q^{-1} + 3 + 4q + 7q^2 + \cdots & 26752 + 1734016q + 46091264q^2 + \cdots \\ 2 + 2q + 6q^2 + \cdots & q^{-1} - 247 - 86241q - 4182736q^2 - \cdots \end{pmatrix}.$$

For the MTCs of rank larger than 2, there are more than one parameter in the resulting matrix $\mathcal{A}$. But we can also compare the first column of the fundamental matrix with the characters of the corresponding known affine Kac-Moody Lie algebra(WZW model) to get the values of the parameters.

The representation ρ of qs_2 is irreducible so there is only one component. The matrices S and T of qs_2 with central charge 1 are
$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and $T = \begin{pmatrix} e^{\frac{23i\pi}{12}} & 0 \\ 0 & e^{\frac{5i\pi}{12}} \end{pmatrix}$.

The canonical basis vectors are $v_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, and $v_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Note that these vectors are the canonical basis vectors corresponding to $\mathbb{X}^{(\xi,1)}$s which determine the order and position of the pole of the representation.

Example 3.3. The characters of the VOA modules with central charge 9 corresponding to the MTC qs_2 of rank 2 .

With central charge 9 and conformal weights 0, 1/4. We have $\lambda_1 = 0 - 9/24 = 15/24 = 5/8$ and $\lambda_2 = 1/4 - 9/24 = -3/24 = -1/8$. So the exponent matrix and characteristic matrix are as followed respectively; $\Lambda = \begin{pmatrix} \frac{5}{8} & 0 \\ 0 & -\frac{1}{8} \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 251 & 26752 \\ 2 & 1 \end{pmatrix}$. After solving for the fundamental matrix using the above $\mathbf{x}$ we get the characters of the VOA modules with central charge 9 corresponding to the MTC qs_2 of rank 2 as followed;

$\text{ch } M^1 = q^{5/8}\left(q^{-1} + 251 + 4872q + 48123q^2 + 335627q^3 + \cdots\right)$
$\text{ch } M^2 = q^{-1/8}\left(2 + 498q + 8750q^2 + 79248q^3 + \cdots\right)$

Next, we will give an example when the representation ρ corresponding to the MTC is reducible.

Example 3.4. The characters of the VOA modules with central charge 10 corresponding to the MTC $qs_2 \otimes qs_2$ of rank 4.

The MTC $qs_2 \otimes qs_2$ is of rank 4 with conformal weights $h_1 = 0, h_2 = 1/4, h_3 = 1/4$, and $h_4 = 1/2$.

We have $h_1' = h_1 - 10/24 = -5/12$, $h_2' = -1/6$, $h_3' = -1/6$, and $h_4' = 1/12$.

After decomposing the representation ρ of the MTC $qs_2 \otimes qs_2$, we have $\rho = \rho_1 \oplus \rho_2$, where ρ_1 is a one dimensional irreducible representation with h_2' forming its exponent matrix Λ_1 and ρ_2 is a three dimensional irreducible representation with h_1', h_3', and h_4' forming its exponent matrix Λ_2. The S_i and T_i matrices are $S_1 = (-1)$ and $S_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$ and $T_1 = (e^{\frac{5i\pi}{3}})$ and

$$T_2 = \begin{pmatrix} e^{\frac{7i\pi}{6}} & 0 & 0 \\ 0 & e^{\frac{5i\pi}{3}} & 0 \\ 0 & 0 & e^{\frac{i\pi}{6}} \end{pmatrix}$$

The canonical basis vectors are $v_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$, and $v_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$.

The linear combination of the v_js is

$$av_1 + bv_2 + dv_3 + ev_4. \tag{3.16}$$

Since the canonical basis vectors determine the order and the position of the pole of the representation, the resulting basis vectors determine the coefficients of the direct sum of the vectors (columns) in the fundamental matrix.

Note that v_2 is the canonical basis vector for ρ_1. v_1, v_3, and v_4 are the canonical basis vectors for ρ_2.

We have the following matrices of the representations ρ_1 and ρ_2:

- The exponent matrices
  $\Lambda_1 = (-1/6)$
  $\Lambda_2 = \mathrm{Diag}(\frac{7}{12}, \frac{5}{6}, \frac{1}{12})$
- The characteristic matrices
  $\mathbf{x}_1 = (4)$
  $$\mathbf{x}_2 = \begin{pmatrix} 190 & 32 & 4928 \\ 512 & -4 & -22528 \\ 20 & -8 & 66 \end{pmatrix}$$
- The fundamental matrices
  $\Xi_1 = q^{\Lambda_1}\left(q^{-1} + 4 - 196870q - 43775016q^2 - 2767606261q^3 - ... \right)$
  $$\Xi_2 = q^{\Lambda_2}\begin{pmatrix} q^{-1}+190+5245q+... & 32+192q+800q^2+... & 4928+896896q+... \\ 512+10240q+... & q^{-1}-4+6q-... & -22528-2547712q-... \\ 20+1160q+... & -8-80q-408q^2-... & q^{-1}+66+86647q+... \end{pmatrix}.$$

Recall that a character of a VOA module is of the form

$$\mathrm{ch}\ M^j = q^{h_j - c/24}\sum_{n\geq 0}\mathrm{dim}M^j_{n+h_j}q^n = q^{h'_j}\sum_{n\geq 0}\mathrm{dim}M^j_{n+h_j}q^n.$$

So $h'_j = h_j - c/24$ determines whether ch M^j (as a vector valued modular form) has a pole or not, i.e., if $h'_j < 0$, then ch M^j has a pole at infinity.

Remark 3.2.

- ch M^1 always has a pole at infinity since $h_1 = 0$. The coefficient of the first term of ch M^1 has to be 1 since it is the dimension of the subspace $V_0 \simeq \mathbb{C}\mathbf{1}$.
- If $h'_j < 0$, $j \neq 1$, then the corresponding basis vector v_j contributes to the pole of ch M^j. So there is a combination of the first column of the fundamental matrix and the other columns which correspond to $h'_j(< 0)$s, i.e., the columns generated by v_js. So the coefficient of v_j in equation (3.16) has to be nonnegative.

In this case, $h'_3 = -1/6$ with the basis vector v_3 contributes to the pole of ch M^2. Since v_3 generates the second column of the fundamental matrix of ρ_2, there is a combination of the first and second columns of the fundamental matrix of ρ_2.

Note that the entry of the fundamental matrix of ρ_1 (which corresponds to h'_2) does not contribute to any pole. Since $5/6 = h'_2 \pmod 1 \neq h'_1 \pmod 1 = 7/12$. So this entry cannot contribute to the pole of ch M^1. It also cannot contribute to the pole of ch M^2, since it will give a pole of order larger than one. That is the q-expansion in the fundamental matrix of ρ_1 is $q^{5/6}\left(q^{-2} + 4q^{-1} - 196870 - 43775016q - 2767606261q^2 - \cdots \right)$. So the values of the coefficients in equation (3.16) are $a = 1$, $b = 0$, $d \geq 0$, and $e = 0$. Hence the characters of a VOA and its modules with central charge 10 corresponding to the MTC $qs_2 \otimes qs_2$ are the combination of the first and second columns in the fundamental matrix of ρ_2 and we have the following theorem.

Theorem 3.5. *The characters of the VOAs and their modules with central charge 10 corresponding to the MTC $qs_2 \otimes qs_2$ have the following forms:*

$$\text{ch } M^1 = q^{7/12}(q^{-1} + (190 + 32d) + (5245 + 192d)q + (62150 + 800d)q^2 + \cdots)$$

$$\text{ch } M^2 = q^{5/6}(dq^{-1} + (512 - 4d) + (10240 + 6d)q + (107520 - 8d)q^2 + \cdots)$$

$$\text{ch } M^3 = q^{5/6}(dq^{-1} + (512 - 4d) + (10240 + 6d)q + (107520 - 8d)q^2 + \cdots)$$

$$\text{ch } M^4 = q^{1/12}((20 - 8d) + (1160 - 80d)q + (19324 - 408d)q^2 + \cdots)$$

where d is a suitable nonnegative integer.

Remark 3.3.

- d has to be a nonnegative integer since it appears as the dimension of the submodule in ch M^i.
- ch M^1 contains the dimension of the corresponding reductive Lie algebra V_1 as the second term. So $190+32d$ is the dimension of a reductive Lie algebra V_1.
- V_1 generates an affine Kac-Moody subVOA $\tilde{V}_1$.
- Some MTCs especially those with reducible representations can give more than one corresponding VOAs.

With further consideration using the notion of affine Kac-Moody Lie algebras associated with the VOAs (cf. [3]) which we will not discuss here, we have found that $d = 0, 1, 2,$ or 3 but only $d = 0$ and $d = 2$ give the characters of the VOA modules in this case.

So for $d = 0$ the characters are

$$\text{ch } M^1 = q^{7/12}(q^{-1} + 190 + 5245q + 62150q^2 + \cdots)$$

$$\text{ch } M^2 = q^{5/6}(512 + 10240q + 107520q^2 + \cdots)$$

$$\text{ch } M^3 = q^{5/6}(512 + 10240q + 107520q^2 + \cdots)$$

$$\text{ch } M^4 = q^{1/12}(20 + 1160q + 19324q^2 + \cdots)$$

Remark 3.4. In some cases, there is no explicit reference for the characters of the corresponding Lie algebras but we can do as the following:

1. Compute the fundamental matrix Ξ_1 of a MTC with central charge c.
2. Then take a tensor product of Ξ_1 and the fundamental matrix corresponding to $E_{8,1}\otimes E_{8,1}$ (the fundamental matrix of the trivial MTC with central charge 16).
3. Compute the fundamental matrix Ξ_2 of the same MTC as in step 1 but with central charge $c + 16$.
4. Compare the first column of the resulting matrix from step 2 with the first column of the fundamental matrix Ξ_2.

In our computation, we use some of the CAS available such as using Magma to decompose the reducible representation into the direct sum of irreducible representations and using Mathematica to compute the fundamental matrices and the characters ch M. With the versions of the software we have used at the time of conducting the computation, there was a limitation in computing certain values. So we could only compute the characters of the VOA modules for only the VOA with central charge up to 16.

The following table contains some results from our computation. We will give only the characters of the VOA modules corresponding to the MTCs from Table 1 from number 1 to 10.

Table 2: The characters ch M^i of the irreducible VOA modules corresponding to the MTCs from Table 1

No.	MTC	n	c	Characters ch M^i
1	t_m	1	8	ch $M^1 = q^{2/3}\left(q^{-1} + 248 + 4124q + 34752q^2 + \cdots\right)$
			16	ch $M^1 = q^{1/3}\left(q^{-1} + 496 + 69752q + 2115008q^2 + \cdots\right)$
2	qs_2	2	1	ch $M^1 = q^{23/24}\left(q^{-1} + 3 + 4q + 7q^2 + 13q^3 + \cdots\right)$ ch $M^2 = q^{5/24}\left(2 + 2q + 6q^2 + 8q^3 + \cdots\right)$
			9	ch $M^1 = q^{5/8}\left(q^{-1} + 251 + 4872q + 48123q^2 + 335627q^3 + \cdots\right)$ ch $M^2 = q^{-1/8}\left(2 + 498q + 8750q^2 + 79248q^3 + \cdots\right)$
3	$\overline{qs_2}$	2	7	ch $M^1 = q^{17/24}\left(q^{-1} + 133 + 1673q + 11914q^2 + 63252q^3 + \cdots\right)$ ch $M^2 = q^{11/24}\left(56 + 968q + 7504q^2 + 42616q^3 + \cdots\right)$
			15	ch $M^1 = q^{3/8}\left(q^{-1} + 381 + 38781q + 1010062q^2 + \cdots\right)$ ch $M^2 = q^{1/8}\left(56 + 14856q + 478512q^2 + 7841752q^3 + \cdots\right)$
4	$LeeYang$	2	14/5	ch $M^1 = q^{53/60}\left(q^{-1} + 14 + 42q + 140q^2 + 350q^3 + \cdots\right)$ ch $M^2 = q^{17/60}\left(7 + 34q + 119q^2 + 322q^3 + \cdots\right)$
			54/5	ch $M^1 = q^{11/20}\left(q^{-1} + 262 + 7638q + 103044q^2 + \cdots\right)$ ch $M^2 = q^{-1/20}\left(7 + 1770q + 37419q^2 + 413314q^3 + \cdots\right)$
5	$\overline{LeeYang}$	2	26/5	ch $M^1 = q^{47/60}\left(q^{-1} + 52 + 377q + 1976q^2 + 7852q^3 + ...\right)$ ch $M^2 = q^{23/60}\left(26 + 299q + 1702q^2 + 7475q^3 + \cdots\right)$
			66/5	ch $M^1 = q^{9/20}\left(q^{-1} + 300 + 17397q + 344672q^2 + \cdots\right)$ ch $M^2 = q^{1/20}\left(26 + 6747q + 183078q^2 + 2566199q^3 + \cdots\right)$
6	qs_3	3	2	ch $M^1 = q^{11/12}\left(q^{-1} + 8 + 17q + 46q^2 + 98q^3 + \cdots\right)$ ch $M^2 = q^{1/4}\left(1 + 3q + 9q^2 + 19q^3 + \cdots\right)$ ch $M^3 = q^{1/4}\left(1 + 3q + 9q^2 + 19q^3 + \cdots\right)$
			10	ch $M^1 = q^{7/12}\left(q^{-1} + 256 + 6125q + 72006q^2 + \cdots\right)$ ch $M^2 = q^{-1/12}\left(1 + 251q + 4877q^2 + 49375q^3 + \cdots\right)$ ch $M^3 = q^{-1/12}\left(1 + 251q + 4877q^2 + 49375q^3 + \cdots\right)$

Continued on next page

Table 2 – *Continued from previous page*

No.	MTC	n	c	Characters ch M^i
7	$\overline{qs_3}$	3	6	ch $M^1 = q^{3/4}\left(q^{-1}+78+729q+4382q^2+19917q^3+\cdots\right)$ ch $M^2 = q^{5/12}\left(1+14q+92q^2+456q^3+\cdots\right)$ ch $M^3 = q^{5/12}\left(1+14q+92q^2+456q^3+\cdots\right)$
			14	ch $M^1 = q^{5/12}\left(q^{-1}+326+24197q+541598q^2+\cdots\right)$ ch $M^2 = q^{1/12}\left(1+262q+7688q^2+115760q^3+\cdots\right)$ ch $M^3 = q^{1/12}\left(1+262q+7688q^2+115760q^3+\cdots\right)$
8	$Ising1$	3	1/2	ch $M^1 = q^{47/48}\left(q^{-1}+q+q^2+2q^3+\cdots\right)$ ch $M^2 = q^{23/48}\left(1+q+q^2+q^3+\cdots\right)$ ch $M^3 = q^{1/24}\left(1+q+q^2+2q^3+\cdots\right)$
			17/2	ch $M^1 = q^{31/48}\left(q^{-1}+(136+112d)+(2669+1456d)q+\cdots\right)$ ch $M^2 = q^{7/48}\left((17-16d)+(697-448d)q+\cdots\right)$ ch $M^3 = q^{17/24}\left(dq^{-1}+(256-7d)+(4352+21d)q++\cdots\right)$ where d is a non negative integer
9	$\overline{Ising1}$	3	15/2	ch $M^1 = q^{11/16}\left(q^{-1}+105+1590q+12160q^2+\cdots\right)$ ch $M^2 = q^{3/16}\left(15+470q+4593q^2+30075q^3+\cdots\right)$ ch $M^3 = q^{5/8}\left(128+1920q+15360q^2+88960q^3+\cdots\right)$
			31/2	ch $M^1 = q^{17/48}\left(q^{-1}+(248+7d)+(31124+42d)q++\cdots\right)$ ch $M^2 = q^{41/48}\left(dq^{-1}+(3875+21d)+(181753+84d)q+\cdots\right)$ ch $M^3 = q^{7/24}\left((248-8d)+(34504-56d)q+\cdots\right)$ where d is a non negative integer
10	$Ising2$	3	3/2	ch $M^1 = q^{15/16}\left(q^{-1}+3+9q+15q^2+30q^3+\cdots\right)$ ch $M^2 = q^{7/16}\left(3+4q+12q^2+21q^3+\cdots\right)$ ch $M^3 = q^{1/8}\left(2+6q+12q^2+26q^3+\cdots\right)$
			19/2	ch $M^1 = q^{29/48}\left(q^{-1}+(171+40d)+(4237+320d)q+\cdots\right)$ ch $M^2 = q^{5/48}\left((19-8d)+(988q-120d)q+\cdots\right)$ ch $M^3 = q^{19/24}\left(dq^{-1}+(512-5d)+(9728q+10d)q+\cdots\right)$ where d is a non negative integer

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Nakorn Junla
Mahidol Wittayanusorn School
364 Moo 5 Salaya Phuttamonthon
Nakhonpatom 73170
Email: `nakorn@mwit.ac.th`