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$e-$filters in Stone Almost Distributive Lattices

N. Rafi[∗], Ravi Kumar Bandaru
and G.C. Rao

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Abstract: The concept of $e-$ filters is introduced in a Stone Almost Distributive lattice and element wise characterization is developed for $e-$ filters. Several properties are derived on $e-$ filters with the help of maximal filters. It is also proved that the set of all $e-$ filters forms a complete distributive lattice.

Keywords: Almost Distributive Lattice(ADL), Pseudo-complemented ADL, Stone ADL, Disjunctive ADL, maximal filter, $e-$ filter, prime ideal, dense element

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1 Introduction

After Booles axiomatization of two valued propositional calculus as a Boolean algebra, a number of generalizations both ring theoretically and lattice theoretically have come into being. The concept of an Almost Distributive Lattice(ADL) was introduced by Swamy and Rao [10] as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper, the concept of an ideal in an

[∗] *corresponding author*

ADL was introduced analogous to that in a distributive lattice and it was observed that the set $PI(L)$ of all principal ideals of L forms a distributive lattice. This provided a path to extend many existing concepts of lattice theory to the class of ADLs. With this motivation, Swamy, Rao and Nanaji[11] introduced the concept of pseudo-complementation on an ADL. They observed that unlike in a distributive lattice, an ADL L can have more than one pseudo-complementation. If $*$, $\perp$ are two pseudo-complementations on L, it was observed that $x^* \vee x^{**}$ is maximal, for all $x \in L$ if and only if $x^{\perp} \vee x^{\perp\perp}$ is maximal, for all $x \in L$. With this motivation, in [12], the concept of a Stone ADL was introduced as an ADL with a pseudo-complementation $*$ satisfying the condition $x^* \vee x^{**}$ is maximal, for all $x \in L$. They studied the properties of pseudo-complemented ADLs and characterized Stone ADLs algebraically, topologically and by means of prime ideals. In [9], Sambasiva Rao introduced $e-$filters in an $MS-$algebra and proved some related properties. In this paper, $e-$filters are extended to an ADLs with Stone property. Though many results look similar, the proofs are not similar because we do not have the properties like commutativity of $\vee$, commutativity of $\wedge$ and the right distributivity of $\vee$ over $\wedge$ in an ADL. We introduced the definition $e-$filter in a Stone ADL in tems of the annihilator ideals. It is observed that set of all dense elements will be an $e-$filter. In addition to this, we obtained the class of all $e-$filters forms a complete distributive lattice. $e-$filter is characterized in element wise. It is established that every maximal filter of Stone ADL is always an $e-$filter. It is also observed that every minimal prime filter which contains a given $e-$filter is an $e-$filter. Finally, we proved some fruitful results on $e-$filters in terms of maximal filters.

2 Preliminaries

First, we recall certain definitions and properties of ADLs, Pseudo-complemented ADLs and Stone ADLs that are required in the paper. We begin with ADL definition as follows.

Definition 2.1. [10] An Almost Distributive Lattice with zero or simply ADL is an algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ satisfying:

1. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3. $(x \vee y) \wedge y = y$

4. $(x \vee y) \wedge x = x$
5. $x \vee (x \wedge y) = x$
6. $0 \wedge x = 0$
7. $x \vee 0 = x,$ for all $x, y, z \in L$.

Example 2.2. Every non-empty set X can be regarded as an ADL as follows. Let $x_0 \in X$. Define the binary operations $\vee, \wedge$ on X by

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \qquad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL (where x_0 is the zero) and is called a discrete ADL.

If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b \in L$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on L.

Theorem 2.3. *[10] If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in L$, we have the following:*

(1). $a \vee b = a \Leftrightarrow a \wedge b = b$
(2). $a \vee b = b \Leftrightarrow a \wedge b = a$
(3). $\wedge$ *is associative in* L
(4). $a \wedge b \wedge c = b \wedge a \wedge c$
(5). $(a \vee b) \wedge c = (b \vee a) \wedge c$
(6). $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
(7). $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(8). $a \wedge (a \vee b) = a, \quad (a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$
(9). $a \leq a \vee b$ *and* $a \wedge b \leq b$
(10). $a \wedge a = a$ *and* $a \vee a = a$
(11). $0 \vee a = a$ *and* $a \wedge 0 = 0$
(12). *If* $a \leq c, \quad b \leq c$ *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$
(13). $a \vee b = (a \vee b) \vee a$.

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL L a distributive lattice. That is

Theorem 2.4. *[10] Let $(L, \vee, \wedge, 0)$ be an ADL with 0. Then the following are equivalent:*

1). $(L, \vee, \wedge, 0)$ *is a distributive lattice*
2). $a \vee b = b \vee a$*, for all* $a, b \in L$
3). $a \wedge b = b \wedge a$*, for all* $a, b \in L$
4). $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$*, for all* $a, b, c \in L$.

As usual, an element $m \in L$ is called maximal if it is a maximal element in the partially ordered set $(L, \leq)$. That is, for any $a \in L$, $m \leq a \Rightarrow m = a$.

Theorem 2.5. *[10] Let L be an ADL and $m \in L$. Then the following are equivalent:*

1). m *is maximal with respect to* $\leq$
2). $m \vee a = m$*, for all* $a \in L$
3). $m \wedge a = a$*, for all* $a \in L$
4). $a \vee m$ *is maximal, for all* $a \in L$.

As in distributive lattices [1, 3], a non-empty subset I of an ADL L is called an ideal of L if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in L$. Also, a non-empty subset F of L is said to be a filter of L if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in L$.

The set $I(L)$ of all ideals of L is a bounded distributive lattice with least element $\{0\}$ and greatest element L under set inclusion in which, for any $I, J \in I(L)$, $I \cap J$ is the infimum of I and J while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. A proper ideal P of L is called a prime ideal if, for any $x, y \in L$, $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. A proper ideal M of L is said to be maximal if it is not properly contained in any proper ideal of L. It can be observed that every maximal ideal of L is a prime ideal. Every proper ideal of L is contained in a maximal ideal. For any subset S of L the smallest ideal containing S is given by $(S] := \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in L \text{ and } n \in N\}$. If $S = \{s\}$, we write $(s]$ instead of $(S]$. Similarly, for any $S \subseteq L$, $[S) := \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in L \text{ and } n \in N\}$. If $S = \{s\}$, we write $[s)$ instead of $[S)$.

Theorem 2.6. *[10] For any x, y in L the following are equivalent:*

1). $(x] \subseteq (y]$
2). $y \wedge x = x$

3). $y \vee x = y$
4). $[y) \subseteq [x)$.

For any $x, y \in L$, it can be verified that $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \wedge y]$. Hence the set $PI(L)$ of all principal ideals of L is a sublattice of the distributive lattice $I(L)$ of ideals of L.

Theorem 2.7 ([5])**.** *Let I be an ideal and F a filter of L such that $I \cap F = \emptyset$. Then there exists a prime ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$.*

For any $A \subseteq L$, $Ann\{A\} = \{x \in L \mid a \wedge x = 0$ for all $a \in A\}$ is an ideal of L. We write $Ann\{(a]\}$ for $Ann\{a\}$. Then clearly $Ann\{(0]\} = L$ and $Ann\{L\} = (0]$.

Definition 2.8. Let L be an ADL and $x \in L$. Then define $Ann\{x\} = \{y \in L \mid x \wedge y = 0\}$. Clearly, $Ann\{x\}$ is an ideal in L and hence an annihilator ideal.

Annihilators have many important properties. We give some of them in the following lemma which can be proved directly.

Lemma 2.9. *Let L be an ADL and for any $x, y \in L$. Then we have:*

(1) $x \leq y \Rightarrow Ann\{y\} \subseteq Ann\{x\}$

(2) $Ann\{(x \wedge y)\} = Ann\{(y \wedge x)\}$

(3) $Ann\{(x \vee y)\} = Ann\{(y \vee x)\}$

(4) $Ann\{(x \vee y)\} = Ann\{x\} \cap Ann\{y\}$

(5) $Ann\{x\} \vee Ann\{y\} \subseteq Ann\{(x \wedge y)\}$.

Definition 2.10 ([11])**.** Let $(L, \vee, \wedge, 0)$ be an ADL. Then a unary operation $a \longrightarrow a^*$ on L is called a pseudo-complementation on L if, for any $a, b \in L$, it satisfies the following conditions:

(1) $a \wedge b = 0 \Rightarrow a^* \wedge b = b$

(2) $a \wedge a^* = 0$

(3) $(a \vee b)^* = a^* \wedge b^*$

Then $(L, \vee, \wedge, ^*, 0)$ is called a pseudo-complemented ADL.

Here, the unary operation $*$ is called a pseudo-complementation on L and a^* is called a pseudo-complement of a in L. An element a of a pseudo-complemnted

ADL L is called a dense element if $a^* = 0$. Let us denote the set of all dense elements of L by D.

Now we list some results of pseudo-complementation.

Theorem 2.11 ([11])**.** *Let L be an ADL and $*$, a pseudo-complementation on L. Then, for any $a, b \in L$, we have the following:*

(1) 0^* *is maximal*

(2) If a is maximal, then $a^* = 0$

(3) $0^{**} = 0$

(4) $a^{**} \wedge a = a$

(5) $a^{***} = a^*$

(6) $a \leq b \Rightarrow b^* \leq a^*$

(7) $a^* \wedge b^* = b^* \wedge a^*$

(8) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

Definition 2.12 ([12])**.** Let L be an ADL and $*$ a pseudo-complementation on L.Then L is called Stone ADL if, for any $x \in L$, $x^* \vee x^{**} = 0^*$.

Lemma 2.13 ([12])**.** *Let L be a Stone ADL and $a, b \in L$. Then the following conditions hold:*

(1) $0^* \wedge a = a$ *and* $0^* \vee a = 0^*$

(2) $(a \wedge b)^* = a^* \vee b^*$.

3 $e-$filters in a Stone ADLs

In [9], M.S Rao introduced the concept of $e-$filters in MS-algebras and studied their properties. In this paper, we extend this concept of $e-$filters to a Stone ADL, analogously and characterized in element wise. Some basic properties of $e-$filters are observed in terms of maximal filters. We proved that every maximal filter of Stone ADL is always an $e-$filter and also observed that every minimal prime filter containing a given $e-$filter is an $e-$filter. Finally, for any filter F of a Stone ADL

L, we derived that $\mathfrak{F}^{\circ}(F) = \bigcap\{\mathfrak{F}(M) \mid M$ is a maximal filter and $F \subseteq M\}$.

Now we begin with the following definition.

Definition 3.1. For any filter F of a Stone ADL L, define an extension of F as the set $F^e = \{x \in L \mid x^* \in Ann\{a\},$ for some $a \in F\}$.

We prove the following result.

Lemma 3.2. *Let L be a Stone ADL. For any two filters F and G of L, we have the following:*
(1). *F^e is a filter of L*
(2). $F \subseteq F^e$
(3). $F \subseteq G \Rightarrow F^e \subseteq G^e$
(4). $(F \cap G)^e = F^e \cap G^e$
(5). $(F^e)^e = F^e$.

Proof. (1). Let m be any maximal element of L. Clearly, $m \in F^e$ and hence $F^e \neq \emptyset$. Let $x, y \in F^e$. Then, $x^* \in Ann\{a\}$ and $y^* \in Ann\{b\}$, for some $a, b \in F$. That implies $x^* \wedge a = 0$ and $y^* \wedge b = 0$. Now, $(x \wedge y)^* \wedge a \wedge b = (x^* \vee y^*) \wedge a \wedge b = (x^* \wedge a \wedge b) \vee (y^* \wedge a \wedge b) = 0$. Therefore $(x \wedge y)^* \in Ann\{a \wedge b\}$. Since $a \wedge b \in F$, we get that $x \wedge y \in F^e$. Let $x \in F^e$ and $r \in L$. Then $x^* \in Ann\{a\}$, for some $a \in L$. That implies $x^* \wedge a = 0$. Now $(r \vee x)^* \wedge a = r^* \wedge x^* \wedge a = 0$. Therefore $(r \vee x)^* \in Ann\{a\}$ and hence $r \vee x \in F^e$. Thus F^e is a filter of L.
(2). Let $x \in F$. Clearly, we have $x^* \wedge x = 0$ and hence $x^* \in Ann\{x\}$. Therefore $x \in F^e$. Thus $F \subseteq F^e$.
(3). Suppose $F \subseteq G$. Let $x \in F^e$. Then, $x^* \in Ann\{a\}$, for some $a \in F$. Since $F \subseteq G$, we get that $x \in G^e$. Therefore, $F^e \subseteq G^e$.
(4). Clearly, $(F \cap G)^e \subseteq F^e \cap G^e$. We prove that $F^e \cap G^e \subseteq (F \cap G)^e$. Let $x \in F^e \cap G^e$. Then, $x^* \in Ann\{a\}$ and $x^* \in Ann\{b\}$, for some $a \in F$ and $b \in G$. That implies, $x^* \in Ann\{a\} \cap Ann\{b\} = Ann\{a \vee b\}$. Since $a \vee b \in F \cap G$, we get that $x \in (F \cap G)^e$ and hence $F^e \cap G^e \subseteq (F \cap G)^e$. Therefore $(F \cap G)^e = F^e \cap G^e$.
(5). By condition (2), we get that $F^e \subseteq (F^e)^e$. Let $x \in (F^e)^e$. Then $x^* \in Ann\{a\}$, for some $a \in F^e$. That implies $x^* \wedge a = 0$ and hence $a^* \wedge x^* = x^*$. Since $a \in F^e$, we get that $a^* \in Ann\{b\}$, for some $b \in F$ and hence $a^* \wedge b = 0$. Now $x^* \wedge b = a^* \wedge x^* \wedge b = 0$. Then $x^* \in Ann\{b\}$. Therefore $x \in F^e$ and hence $(F^e)^e \subseteq F^e$. Thus $(F^e)^e = F^e$.

□

Now, we give the definition of $e-$filter in the following.

Definition 3.3. A filter F of a Stone ADL L is called an $e-$filter of L if $F = F^e$.

We prove the following result.

Lemma 3.4. *Let L be a Stone ADL L. Then D is the smallest $e-$filter of L.*

Proof. Clearly, D is a filter of L and $D \subseteq D^e$. Let $x \in D^e$. Then $x^* \in Ann\{a\}$, for some $a \in D$. That implies that $x^* \wedge a = 0$ and hence $a^* \wedge x^* = x^*$. Therefore $x^* = 0$, since a is a dense element of L. That implies $x \in D$ and hence $D = D^e$. Thus D is an $e-$filter of L. Suppose G is any $e-$filter of L. Let $x \in D$. Then $x^* = 0$. Since $x^* \wedge 0^* = 0$ and $0^* \in G$, we get that $x \in G^e = G$. Therefore $D \subseteq G$. Hence D is the smallest $e-$filter of L. $\square$

From Lemma 3.2, we can observe that the intersection of two e-filters of a Stone ADL is again an e-filter. But, in general, the supremum of two e-filters need not be an e-filter. However, in the following, we obtain the class $\mathfrak{F}^e(L)$ of all e-filters of L that is a distributive lattice.

Theorem 3.5. *Let L be a Stone ADL. Then the class $\mathfrak{F}^e(L)$ of all $e-$filters forms a complete distributive lattice on its own.*

Proof. For any two $e-$filters F, G of L, define the ordering $\leq$ on $\mathfrak{F}^e(L)$ such that $F \leq G \Leftrightarrow F \subseteq G$. Then clearly $(\mathfrak{F}^e(L), \leq)$ is a partially ordered set. Now, consider $F \cap G = (F \cap G)^e$ and $F \sqcup G = (F \vee G)^e$. Then clearly, $(F \cap G)^e$ is the infimum of both F^e and G^e in $\mathfrak{F}^e(L)$. Clearly, $(F \vee G)^e$ is the upper bound for both F and G. Suppose that H is any $e-$filter of L such that $F \subseteq H$ and $G \subseteq H$. Let $x \in (F \vee G)^e$. Then $x^* \in Ann\{a\}$, for some $a \in F \vee G$. Then $x^* \wedge a = 0$. Since $a \in F \vee G$, we can take $a = f \wedge g$, for some $f \in F$ and $g \in G$. Since $F \subseteq H$ and $G \subseteq H$, we get that $a \in H$ and hence $x \in H^e$. Since H is an $e-$filter of L, we get $x \in H$. Therefore, $(F \vee G)^e$ is the supremum for both F and G in $\mathfrak{F}^e(L)$. Hence, $(\mathfrak{F}^e(L), \cap, \sqcup, D, L)$ is a bounded lattice. By the extension of the property (4) of lemma 3.2, $(\mathfrak{F}^e(L), \cap, \sqcup, D, L)$ is a complete lattice. Now for any $F, G, H \in \mathfrak{F}^e(L)$, we obtain $(F^e \sqcup G^e) \cap (F^e \sqcup H^e) = (F \vee G)^e \cap (F \vee H)^e = \{(F \vee G) \cap (F \vee H)\}^e = \{F \vee (G \cap H)\}^e = F^e \sqcup (G \cap H)^e = F^e \sqcup (G^e \cap H^e)$. Therefore, $(\mathfrak{F}^e(L), \cap, \sqcup, D, L)$ is bounded and complete distributive lattice. $\square$

The following result is a direct consequence of the above.

Corollary 3.6. *Every maximal e−filter of* $\mathfrak{F}^e(L)$ *is a prime e−filter.*

In the following we characterize $e-$filter element wise.

Theorem 3.7. *Let* F *be a filter of a Stone ADL* L*. Then, the following are equivalent:*
(1). F *is an* $e-$*filter*
(2). $x^{**} \in F$ *implies* $x \in F$, *for all* $x \in L$
(3). *For* $x, y \in L$, $x^* = y^*$ *and* $x \in F$ *imply that* $y \in F$.

Proof. $(1) \Rightarrow (2)$: Assume that F is an $e-$filter of L. Let x be any element of L with $x^{**} \in F$. Since F is an $e-$filter of L, we get $x^{**} \in F^e$. Then, $(x^{**})^* \in Ann\{a\}$, for some $a \in F$. That implies that $x^* \in Ann\{a\}$ and hence $x \in F^e$. Therefore $x \in F$.
$(2) \Rightarrow (3)$: Assume the condition (2). Let $x, y \in L$ with $x^* = y^*$ and $x \in F$. Then, $x^{**} = y^{**}$. Since $x^{**} = x^{**} \vee x \in F$, we get $y^{**} \in F$. By our assumption we get that $y \in F$.
$(3) \Rightarrow (1)$: Assume the condition (3). We have $F \subseteq F^e$. It is enough to prove that $F^e \subseteq F$. Let $x \in F^e$. Then, $x^* \in Ann\{a\}$, for some $a \in F$. Hence, $x^* \wedge a = 0$ and hence $(a \vee x)^* = a^* \wedge x^* = x^*$ Since $a \in F$ and F is a filter of L, we get that $a \vee x \in F$. By our assumption we get $x \in F$. Therefore $F^e \subseteq F$. Hence $F = F^e$. Thus F is an $e-$filter of L. □

Proposition 3.8. *Every maximal filter of a Stone ADL is an* $e-$*filter.*

Proof. Let M be a maximal filter of L. We prove that M is an $e-$filter of L. It is enough to prove that condition (2) of theorem 3.7. Let $x^{**} \in M$. Suppose $x \notin M$. Then, $M \vee [x) = L$ and hence $0 = a \wedge x$, for some $a \in M$. That implies $0 = 0^{**} = (a \wedge x)^{**} = a^{**} \wedge x^{**}$. Since M is a filter of L and $a \in M$, we get $a^{**} \in M$, and hence $a^{**} \wedge x^{**} \in M$. Therefore $0 \in M$, Which is a contradiction. Hence $x \in M$. Thus M is an $e-$filter of L. □

We observed that that every maximal filter of Stone ADL is a prime $e-$filter. Now the following result can be easy to verify.

Lemma 3.9. *Let* L *be a Stone ADL. If* $a^* = b^*$ *then* $(a \vee c)^* = (b \vee c)^*$, $(a \wedge c)^* = (b \wedge c)^*$ *and* $(a)^* = (a \vee b)^*$, *for all* $a, b, c \in L$.

Theorem 3.10. *Let* L *be a Stone ADL. If* P *is minimal prime filter of* L *containing a given* $e-$*filter, then,* P *is an* $e-$*filter of* L.

Proof. Let F be an $e-$filter of L and P, a minimal in the class of all prime filter of L such that $F \subseteq P$. We have to prove that P is an $e-$filter of L. Suppose that P is not an $e-$filter. Then by the theorem 3.7(3), there exists elements $x, y \in L$ such that $x^* = y^*, x \in P$ and $y \notin P$. Take $I = (L \setminus P) \vee (x \vee y]$. We prove that $I \cap F = \emptyset$. Suppose $I \cap F \neq \emptyset$. Choose $a \in I \cap F$. Since $a \in I$, we get $a = r \vee s$, for some $r \in L \setminus P$ and $s \in (x \vee y]$. Since $s \in (x \vee y]$, we obtain $(x \vee y) \wedge s = s$. Now, $r \vee s = r \vee [(x \vee y) \wedge s] = (r \vee x \vee y) \wedge (r \vee s)$. That implies $a = (r \vee x \vee y) \wedge a$ and hence $r \vee x \vee y = (r \vee x \vee y) \vee a$. Since $a \in F$, we get that $r \vee x \vee y \in F$. Since $x^* = y^*$, by the above lemma, we obtain $(r \vee y)^* = r^* \wedge y^* = r^* \wedge x^* \wedge y^* = (r \vee x \vee y)^*$. As F is an $e-$filter and $r \vee x \vee y \in F$, we get $r \vee y \in F$ and hence $r \vee y \in P$. Since P is a prime filter, we get that either $r \in P$ or $y \in P$. Since $r \notin P$ we get $y \in P$, which is a contradiction to $y \notin P$. Therefore $I \cap F = \emptyset$. So that by Zorn's lemma, there exists a prime filter Q, such that $I \cap Q = \emptyset$ and $F \subseteq Q$. As $I \cap Q = \emptyset$, we obtain $Q \subseteq P$. Since $x \in P$ and P is a filter, we get $x \vee y \in P$ but $x \vee y \notin Q$. That implies $Q \subset P$. Therefore P is not a minimal in the class of all prime filters containing F, which is a contradiction. Therefore, P is an $e-$filter of L. □

The following definition is taken from [8].

Definition 3.11. An ADL L is said to be a disjunctive ADL if for any $x, y \in L$, $Ann\{x\} = Ann\{y\}$ implies $x = y$.

Theorem 3.12. *Let L be a Stone ADL. If L is a disjunctive ADL, then every filter of L is an $e-$filter.*

Proof. Let L be a disjunctive ADL and F be any filter of L. Clearly, we have $F \subseteq F^e$. Let $x \in F^e$. Then $x^* \in Ann\{a\}$, for some $a \in F$. That implies $x^* \wedge a = 0$ and hence $a^* \wedge x^* = x^*$. That implies $(a \vee x)^{**} = x^{**}$. Since L is a disjunctive ADL, we get $a \vee x = x$. Since $a \in F$, we get $x \in F$. Therefore $F = F^e$. Hence F is an $e-$filter of L. □

We have the following result.

Theorem 3.13. *For any prime filter P of a Stone ADL L, the set $\mathfrak{F}(P) = \{x \in L \mid x^* \notin P\}$ is an $e-$filter of L.*

Proof. Let m be any maximal element of L and P, any prime filter of L. Clearly, $m^* \notin P$ and we get that $m \in \mathfrak{F}(P)$. Let $x, y \in \mathfrak{F}(P)$. Then $x^* \notin P$ and $y^* \notin P$. Since P is a prime filter, we get that $(x \wedge y)^* = x^* \vee y^* \notin P$. Therefore $x \wedge y \in \mathfrak{F}(P)$.

Let $x \in \mathfrak{F}(P)$ and $r \in L$. Then $x^* \notin P$. We prove that $r \vee x \in \mathfrak{F}(P)$. Suppose $(r \vee x)^* \in P$. Then $r^* \wedge x^* \in P$ and hence $x^* \in P$, which is a contradiction. Therefore $r \vee x \in \mathfrak{F}(P)$. Thus $\mathfrak{F}(P)$ is a filter of L. Clearly, we have $\mathfrak{F}(P) \subseteq (\mathfrak{F}(P))^e$. Conversely, let $x \in (\mathfrak{F}(P))^e$. Then $x^* \in Ann\{a\}$, for some $a \in \mathfrak{F}(P)$. That implies $x^* \wedge a = 0$ and hence $a^* \wedge x^* = x^*$. Since $a \in \mathfrak{F}(P)$, we have $a^* \notin P$ and hence $a^* \in L \setminus P$. That implies that $x^* \in L \setminus P$ and hence $x^* \notin P$. Therefore $x \in \mathfrak{F}(P)$. Thus $\mathfrak{F}(P)$ is an $e-$filter of L. □

Corollary 3.14. *For any maximal filter M of a Stone ADL L, $\mathfrak{F}(M)$ is an $e-$filter of L.*

Theorem 3.15. *For any filter F of a Stone ADL L, the set $\mathfrak{F}^\circ(F) = \{x \in L \mid [x^*) \vee F = L\}$ is an $e-$filter of L.*

Proof. Let F be a filter of L and m be any maximal element of L. Now $[m^*) \vee F = [0) \vee F = L \vee F = L$ and hence $m \in \mathfrak{F}^\circ(F)$. Therefore $\mathfrak{F}^\circ(F) \neq \emptyset$. Let $x, y \in \mathfrak{F}^\circ(F)$. Then $[x^*) \vee F = L = [y^*) \vee F$. Now, $[(x \wedge y)^*) \vee F = \{[x^*) \cap [y^*)\} \vee F = \{[x^*) \vee F\} \cap \{[y^*) \vee F\} = L$. Hence $x \wedge y \in \mathfrak{F}^\circ(F)$. Let $x \in \mathfrak{F}^\circ(F)$ and $r \in L$. Then, $[x^*) \vee F = L$. Now, $[(r \vee x)^*) \vee F = [r^* \wedge x^*) \vee F = [r^*) \vee [x^*) \vee F = [r^*) \vee L = L$. Therefore $r \vee x \in \mathfrak{F}^\circ(F)$ and hence $\mathfrak{F}^\circ(F)$ is a filter of L. Clearly, we have $\mathfrak{F}^\circ(F) \subseteq (\mathfrak{F}^\circ(F))^e$. Let $x \in (\mathfrak{F}^\circ(F))^e$. Then $x^* \in Ann\{a\}$, for some $a \in \mathfrak{F}^\circ(F)$. That implies $x^* \wedge a = 0$ and $[a^*) \vee F = L$. Now $[x^*) \vee F = [a^* \wedge x^*) \vee F = [x^*) \vee [a^*) \vee F = [x^*) \vee L = L$. Therefore $x \in \mathfrak{F}^\circ(F)$ and hence $\mathfrak{F}^\circ(F)$ is an $e-$filter of L. □

Now, we conclude this paper with the following theorem.

Theorem 3.16. *Let F be a filter of a Stone ADL L. Then, we have $\mathfrak{F}^\circ(F) = \bigcap\{\mathfrak{F}(M) \mid M$ is a maximal filter and $F \subseteq M\}$.*

Proof. Take $\mathfrak{M} = \bigcap\{\mathfrak{F}(M) \mid M$ is a maximal filter and $F \subseteq M\}$. Let M be a maximal filter of L with $F \subseteq M$ and $x \in \mathfrak{F}^\circ(F)$ Then, $[x^*) \vee F = L$ and hence $[x^*) \vee M = L$. That implies $x^* \wedge a = 0$, for some $a \in M$ and hence $x^* \notin M$. Therefore $x \in \mathfrak{F}(M)$. Hence $\mathfrak{F}^\circ(F) \subseteq \mathfrak{M}$. Conversely, let $x \in \mathfrak{M}$. Then, $x \in \mathfrak{F}(M)$, for all maximal filters M of L such that $F \subseteq M$. Then $x^* \notin M$. We prove that $[x^*) \vee F = L$. Suppose $[x^*) \vee F \neq L$. Then, there exists a maximal filter M_0 of L such that $[x^*) \vee F \subseteq M_0$. Hence, $x^* \in M_0$ and $F \subseteq M_0$ which is a contradiction. Thus, $[x^*) \vee F = L$. Therefore, $x \in \mathfrak{F}^\circ(F)$. Thus $\mathfrak{M} \subseteq \mathfrak{F}^\circ(F)$. Hence $\mathfrak{F}^\circ(F) = \bigcap\{\mathfrak{F}(M) \mid M$ is a maximal filter and $F \subseteq M\}$. □

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N. Rafi
Department of Mathematics
Bapatla Engineering College
Bapatla, Andhra Pradesh, INDIA-522 101
Email: rafimaths@gmail.com

Ravi Kumar Bandaru
Department of Engg. Mathematics
GITAM University, Hyderabad Campus
Andhra Pradesh, India-502 329
Email: ravimaths83@gmail.com

G.C. Rao
Department of Mathematics
Andhra University, Visakhapatnam
Andhra Pradesh, India-530 003
Email: gcraomaths@yahoo.co.in