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On maps preserving strongly zero-products

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Abstract: The notion of a strongly zero-product preserving map on normed algebras recently was introduced by the author. This notion is a generalization of the well-known notion "zero-product preserving map." We give a characterization of strongly zero-product preserving maps on normed algebras and also by giving some illustrative and interesting examples. We show that this notion is completely different from the notion of zero-product preserving maps. Also we show that the direct product of two strongly zero-product preserving maps is again a strongly zero-product preserving map. But the tensor product of them need not be a strongly zero-product preserving map. Finally we show that every $*$-preserving linear map from a normed $*$-algebra into a C^*-algebra that strongly preserves zero-products is necessarily continuous.

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1 Introduction

A linear map $\varphi : A \longrightarrow B$ between two algebras A and B over a field F is said to be a zero-product preserving map if $\varphi(a)\varphi(c) = 0$ whenever $ac = 0$, $(a, c \in A)$. For some good references in the field of zero-product preserving maps we refer the reader to [1, 2].

Let A and B be two normed algebras over a field F. We say that a linear map $\varphi : A \longrightarrow B$ is a strongly zero-product preserving map if, for any two sequences $\{a_n\}_n$, $\{c_n\}_n$ in A, $\varphi(a_n)\varphi(c_n) \longrightarrow 0$ whenever $a_n c_n \longrightarrow 0$. Clearly every strongly zero-product preserving map is a zero-product preserving map. But the converse is not the case (see Example 2.4). So the concept of strongly zero-product preserving maps is a generalization of the concept of zero-product preserving maps.

Strongly zero-product preserving maps on a certain class of normed algebras is characterized in [4].

For the normed algebras A and B over a field F, we will denote by $A \otimes B$, the algebraic tensor product of A and B. It is well-known that $A \otimes B$ is a normed algebra with the projective cross norm, given by

$$\|u\| = \inf\Big\{ \sum_{k=1}^{k=n} \|a_k\|\|b_k\|, \quad u = \sum_{k=1}^{k=n} a_k \otimes b_k, \quad a_k \in A,\ b_k \in B,\ n \in \mathbb{N} \Big\},$$

for all $u \in A \otimes B$.

The purpose of the present paper is to introduce and characterize the strongly zero-product preserving maps on normed algebras as a generalization of the well-known notion "zero-product preserving maps" on normed algebras. Also we give some examples to show that these notions are completely different. We prove that the composition and the direct product of two strongly zero-product preserving maps is again a strongly zero-product preserving map. But the tensor product of them need not be a strongly zero-product preserving map. Finally we show that every $*$-preserving linear map from a normed $*$-algebra into a C^*-algebra that strongly preserves zero-products is necessarily continuous.

2 Definitions and examples

Definition 2.1. Let A and B be two normed algebras over a field F. We say that a linear map $\varphi : A \longrightarrow B$ is a strongly zero-product preserving map if, for any two sequences $\{a_n\}_n$, $\{c_n\}_n$ in A, $\varphi(a_n)\varphi(c_n) \longrightarrow 0$ whenever $a_n c_n \longrightarrow 0$.

Example 2.2. i. Every continuous homomorphism between normed algebras is a strongly zero-product preserving map.

ii. Let $\mathcal{V}$ be an infinite dimensional normed vector space with the basis $\beta = \{e_1, e_2, e_3, ...\}$ such that $\|e_n\| = 1$ for all $n \geqslant 1$. Also let $f \in \mathcal{V}^*$ be a continuous linear functional satisfying, $f(e_1) = 1$ and $f(e_n) = 0$ for all $n \geqslant 2$. So kerf=span$\{e_2, e_3, e_4, ...\}$.
For all $a, b \in \mathcal{V}$, define $a \cdot b = f(a)b$. Clearly $(\mathcal{V}, \cdot)$ is an associative normed algebra (for the basic properties of this algebra see [3]). We will denote it by $\mathcal{V}_f$.
Define the linear map $\varphi : \mathcal{V}_f \longrightarrow \mathcal{V}_f$ such that $\varphi(e_1) = 0$ and $\varphi(e_n) = 2^n e_2$ for all $n \geqslant 2$. Since $f \circ \varphi \equiv 0$, it is obvious that φ is a strongly zero-product preserving map. We show that φ is neither a continuous map nor a homomorphism on $\mathcal{V}_f$. To this end let $a_n = \frac{e_n}{n}$. So $\|a_n\| = \frac{1}{n}\|e_n\| = \frac{1}{n} \longrightarrow 0$. But $\lim_{n\to\infty} \|\varphi(a_n)\| = \lim_{n\to\infty} \frac{2^n}{n}\|e_2\| = \lim_{n\to\infty} \frac{2^n}{n} = \infty$. This shows that φ is not a continuous map. Also $4e_2 = \varphi(e_2) = \varphi(e_1 e_2) \neq \varphi(e_1)\varphi(e_2) = 0$. So φ is not a homomorphism.

The part ii of Example 2.2 shows that strongly zero-product preserving maps are not continuous in general.

Remark 2.3. It is obvious that every strongly zero-product preserving map is a zero-product preserving map. But the converse is not the case in general. The following example shows this fact.

Example 2.4. Let $\mathcal{V}_f$ be the normed algebra described in Example 2.2. Define $\varphi : \mathcal{V}_f \longrightarrow \mathcal{V}_f$ such that $\varphi(e_1) = e_1$ and $\varphi(e_n) = 2^n e_2$ for all $n \geqslant 2$. It is obvious that $\varphi(\text{kerf}) \subseteq \text{kerf}$. So by [4, Theorem 2.2], φ is a zero-product preserving map. We show that φ is not a strongly zero-product preserving map. To this end let $a_n = \frac{e_1}{n}$ and $c_n = e_n$. So $\|a_n c_n\| = \frac{1}{n}\|e_n\| = \frac{1}{n} \longrightarrow 0$. But $\lim_{n\to\infty} \|\varphi(a_n)\varphi(c_n)\| = \lim_{n\to\infty} \frac{2^n}{n}\|e_2\| = \lim_{n\to\infty} \frac{2^n}{n} = \infty$. This shows that φ is not a strongly zero-product preserving map.

3 Main results

In this section we give a characterization of strongly zero-product preserving maps. Also we prove that every $*$-preserving linear map from a normed $*$-algebra into a C^*-algebra that strongly preserves zero-products is necessarily continuous.

Theorem 3.1. *Let A and B be normed algebras. Then a linear map $\varphi : A \longrightarrow B$ is a strongly zero-product preserving map if and only if, there exists $M > 0$ such that*

$$\|\varphi(a)\varphi(c)\| \leqslant M\|ac\|, (a, c \in A).$$

Proof. Seeking a contradiction, suppose that φ is a strongly zero-product preserving map and the desired inequality is not true for all $M > 0$. So for $M = 1$ there exist $a_1,\ c_1 \in A$ such that,

$$\|\varphi(a_1)\varphi(c_1)\| > \|a_1c_1\|.$$

For $M = \frac{2}{\|\varphi(a_1)\varphi(c_1)\|}$ there exist $a_2,\ c_2 \in A$ such that

$$\|\varphi(a_2)\varphi(c_2)\| > \frac{2}{\|\varphi(a_1)\varphi(c_1)\|}\|a_2c_2\|.$$

It follows that $\|\frac{a_2}{\|\varphi(a_2)\varphi(c_2)\|}c_2\| < \frac{\|\varphi(a_1)\varphi(c_1)\|}{2}$. A similar argument can be applied to show that, for $M = \frac{n}{\|\varphi(a_1)\varphi(c_1)\|}$, there exist $a_n,\ c_n \in A$ such that $\|\frac{a_n}{\|\varphi(a_n)\varphi(c_n)\|}c_n\| < \frac{\|\varphi(a_1)\varphi(c_1)\|}{n}$. Let $a'_n = \frac{a_n}{\|\varphi(a_n)\varphi(c_n)\|}$ and $c'_n = c_n$. As $a'_nc'_n \longrightarrow 0$, it follows that $\varphi(a'_n)\varphi(c'_n) \longrightarrow 0$. That is a contradiction.

The converse is obvious. □

Definition 3.2. Let A and B be two $*$-algebras. We say that a linear map $\varphi : A \longrightarrow B$ is $*$-preserving if, $\varphi(a^*) = \varphi(a)^*$, $(a \in A)$.

Obviously every $*$-homomorphism between A and B is a $*$-preserving map.

Proposition 3.3. *Let A be a normed $*$-algebra and B be a C^*-algebra. Also let $\varphi : A \longrightarrow B$ be a $*$-preserving linear map that strongly preserves zero-products. Then φ is continuous.*

Proof. Let $\varphi : A \longrightarrow B$ be a $*$-preserving linear map that strongly preserves zero-products. Then by Theorem 3.1 there exists $M > 0$ such that $\|\varphi(a)\varphi(c)\| \leqslant M\|ac\|, (a,\ c \in A)$. Let $a = c^*$. So $\|\varphi(c)\|^2 = \|\varphi(c)^*\varphi(c)\| = \|\varphi(c^*)\varphi(c)\| \leqslant M\|c^*c\| \leqslant M\|c^*\|\|c\| = M\|c\|^2$. It follows that $\|\varphi(c)\| \leqslant \sqrt{M}\|c\|$. That implies φ is continuous. □

Remark 3.4. i. Let A and B be normed algebras and let B be unital. Then every surjective strongly zero-product preserving map is continuous. Indeed let $\varphi : A \longrightarrow B$ be a surjective strongly zero-product preserving map. So there exists $a \in A$ such that $\varphi(a) = 1_B$. Let $\{a_n\}_n$ be a sequence in A such that $a_n \longrightarrow 0$. So $a_na \longrightarrow 0$. It follows that $\varphi(a_n) = \varphi(a_n)\varphi(a) \longrightarrow 0$.

ii. Let A and B be two unital normed algebras with the units 1_A and 1_B respectively. Also let $\varphi : A \longrightarrow B$ be a strongly zero-product preserving map such that $\varphi(1_A) = 1_B$. Then φ is continuous. Indeed let $\{a_n\}_n$ be a sequence in A such that $a_n \longrightarrow 0$. So $a_n 1_A \longrightarrow 0$. Since φ is a strongly zero-product preserving map we have

$$\varphi(a_n) = \varphi(a_n)\varphi(1_A) \longrightarrow 0.$$

4 Hereditary properties

In this section we show that the direct product and composition of two strongly zero-product preserving maps is again a strongly zero-product preserving map. But the tensor product of them need not be a strongly zero-product preserving map.

Proposition 4.1. *Let A, B, C, D be normed algebras and let $\varphi : A \longrightarrow B$ and $\psi : C \longrightarrow D$ be two strongly zero-product preserving maps. Then $\varphi \oplus \psi : A \oplus C \longrightarrow B \oplus D$ is a strongly zero-product preserving map.*

Proof. As φ and ψ are strongly zero-product preserving maps, there exist $M,\ N > 0$ such that $\|\varphi(a)\varphi(a')\| \leqslant M\|aa'\|$ and $\|\psi(c)\psi(c')\| \leqslant N\|cc'\|$, $(a,\ a' \in A,\ c,\ c' \in C)$. So

$$\begin{aligned}
\|(\varphi \oplus \psi)((a,c))(\varphi \oplus \psi)((a',c'))\| &= \|(\varphi(a), \psi(c))(\varphi(a'), \psi(c'))\| \\
&= \|(\varphi(a)\varphi(a'), \psi(c)\psi(c'))\| \\
&= \|\varphi(a)\varphi(a')\| + \|\psi(c)\psi(c')\| \\
&\leqslant M\|aa'\| + N\|cc'\| \\
&\leqslant M(\|aa'\| + \|cc'\|) + N(\|aa'\| + \|cc'\|) \\
&= (M+N)(\|aa'\| + \|cc'\|) \\
&= (M+N)\|(aa', cc')\| \\
&= (M+N)\|(a,c)(a',c')\|.
\end{aligned}$$

□

Applying Theorem 3.1 shows that $\varphi \oplus \psi$ is a strongly zero-product preserving map.

Proposition 4.2. *Let A, B, C be normed algebras and let $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow C$ be two strongly zero-product preserving maps. Then $\psi \circ \varphi : A \longrightarrow C$ is a strongly zero-product preserving map.*

Proof. As φ and ψ are strongly zero-product preserving maps, there exist $M,\ N > 0$ such that $\|\varphi(a)\varphi(a')\| \leqslant M\|aa'\|$ and $\|\psi(b)\psi(b')\| \leqslant N\|bb'\|$, $(a,\ a' \in A,\ b,\ b' \in B)$. So

$$\begin{aligned} \|(\psi\circ\varphi)(a)(\psi\circ\varphi)(a')\| &= \|\psi(\varphi(a))\psi(\varphi(a'))\| \\ &\leqslant N\|\varphi(a)\varphi(a')\| \\ &\leqslant MN\|aa'\|,\ (a,\ a' \in A). \end{aligned}$$

This shows that $\psi \circ \varphi$ is a strongly zero-product preserving map. □

Remark 4.3. The tensor product of two strongly zero-product preserving maps need not be a strongly zero-product preserving map. Indeed let $\mathcal{A} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$ such that

$$i^2 = j^2 = k^2 = -1$$

and

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

It is well-known that $\mathcal{A}$ is a division algebra over $\mathbb{R}$ that is, an algebra over $\mathbb{R}$ in which every non-zero element is invertible. Define

$$\|a + bi + cj + dk\| = 4\max\{|\mathrm{a}|,\ |\mathrm{b}|,\ |\mathrm{c}|,\ |\mathrm{d}|\}.$$

One can simply verify that $\mathcal{A}$ is a normed algebra over $\mathbb{R}$.

Define $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ by $\varphi(a + bi + cj + dk) = bi$. We show that φ is a strongly zero-product preserving map. For this end let $x = a + bi + cj + dk$ and $y = e + fi + gj + hk$ be two arbitrary elements of $\mathcal{A}$. Let

$$xy = A + Bi + Cj + Dk. \tag{1}$$

By the equation (1), we can conclude that

$$\begin{cases} ea - fb - gc - hd = A \\ fa + eb + hc - gd = B \\ ga - hb + ec + fd = C \\ ha + gb - fc + ed = D. \end{cases} \tag{2}$$

Let $y = e + fi + gj + hk \neq 0$. So $e^2 + f^2 + g^2 + h^2 \neq 0$. Using Cramer's rule to find b from (2), yields

$$\begin{aligned} b = &-\frac{f}{e^2+f^2+g^2+h^2}A + \frac{e}{e^2+f^2+g^2+h^2}B \\ &-\frac{h}{e^2+f^2+g^2+h^2}C + \frac{g}{e^2+f^2+g^2+h^2}D. \end{aligned}$$

So

$$\begin{aligned} bf = &-\frac{f^2}{e^2+f^2+g^2+h^2}A + \frac{fe}{e^2+f^2+g^2+h^2}B \\ &-\frac{fh}{e^2+f^2+g^2+h^2}C + \frac{fg}{e^2+f^2+g^2+h^2}D. \end{aligned}$$

It follows that

$$\begin{aligned} |bf| &\leqslant \frac{|f^2|}{e^2+f^2+g^2+h^2}|A| + \frac{|fe|}{e^2+f^2+g^2+h^2}|B| \\ &+ \frac{|fh|}{e^2+f^2+g^2+h^2}|C| + \frac{|fg|}{e^2+f^2+g^2+h^2}|D| \\ &\leqslant |A| + \frac{1}{2}|B| + \frac{1}{2}|C| + \frac{1}{2}|D|. \end{aligned}$$

So

$$\begin{aligned} \|\varphi(x)\varphi(y)\| &= \|(bi)(fi)\| = \|bf\| = 4|bf| \\ &\leqslant 4|A| + 2(|B| + |C| + |D|) \\ &\leqslant 16 \max\{|\mathrm{A}|, |\mathrm{B}|, |\mathrm{C}|, |\mathrm{D}|\} \\ &= 4\|xy\|. \end{aligned} \tag{3}$$

In the case where $y = e + fi + gj + hk = 0$,

$$\|\varphi(x)\varphi(y)\| = 0 \leqslant 4\|xy\|. \tag{4}$$

So by (3) and (4),

$$\|\varphi(x)\varphi(y)\| \leqslant 4\|xy\|, \ (x, y \in \mathcal{A}).$$

This shows that φ is a strongly zero-product preserving map.

But $\varphi \otimes \varphi : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ is not a strongly zero-product preserving map. Because, $\varphi \otimes \varphi$ is not a zero-product preserving map. Indeed

$$(i \otimes i + j \otimes j)(i \otimes i - j \otimes j) = 0,$$

but

$$\begin{aligned} & \varphi\otimes\varphi(i\otimes i+j\otimes j)\varphi\otimes\varphi(i\otimes i-j\otimes j) \\ = \; & (\varphi(i)\otimes\varphi(i)+\varphi(j)\otimes\varphi(j))(\varphi(i)\otimes\varphi(i)-\varphi(j)\otimes\varphi(j)) \\ = \; & (i\otimes i)(i\otimes i)=1\otimes 1\neq 0. \end{aligned}$$

So $\varphi\otimes\varphi$ is not a strongly zero-product preserving map.

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