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Modified Finite Integration Method by Using Legendre Polynomials for Solving Linear Ordinary Differential Equations

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Abstract: In this paper, we construct a numerical procedure which is called the finite integration method by using the Legendre polynomial. This numerical procedure is for solving the linear ordinary differential equations. That is, we define the solution as a linear combination of the Legendre polynomials and we use the zeros of Legendre polynomial as computational grid points. We implement this procedure with several numerical examples to demonstrate the accuracy of our method comparing to the finite difference method, the traditional finite integration methods and their analytical solutions.

Keywords: Finite integration method, Legendre polynomials

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1 Introduction

Usually, we can explain several phenomina occuring in sciences, engineering and economy by using differential equations. However, under various boundary conditions and the real problem configuration, it is very difficult that these equations

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can be solved for their analytical solution. The numerical methods ease this difficulty and play an important role for finding approximate solutions. Actually, there are many numerical methods available for solving differential equations such as finite difference method (FDM), finite element method (FEM), boundary element method (BEM), etc (see [2]).

In 2013, Wen et al. [6] and Li et al. [4], used both trapezoidal integral algorithm and radial basis functions to develop a new numerical procedure for finding approximate solutions to linear boundary value problems for ordinary and partial differential equations. This is called the finite integration method (FIM). In this method, the finite integration matrix of the first order is obtained by the direct numerical integration, for examples, using trapezoidal [6] and Simpson, Newton-Cotes and Lagrange formula [5]. Based on this finite integration matrix of the first order, any finite integration matrix of other orders for multi-layer integration can be obtained directly by using the matrix of the first order integration. Recently, Duangpan [3] modified the traditional FIM by using the Chebyshev polynomials to construct the finite integration of the first order instead. In the same situation, his modified method obtained more accuracy comparing to the traditional FIM.

In this paper, we turn our attention to construct the FIM by using the Legendre polynomial instead. That is, we define the approximate solution as a linear combination of the Legendre polynomials. We replace the solution domain with a finite number of points, known as grid points, and obtain the solution at these points. The grid points is generated by the zeros of Legendre polynomial of certain degree. The finite integration matrix of the first and higher orders are constructed. Finally, we implement this method with several numerical examples to demonstrate the accuracy of our modified FIM comparing to the FDM, the traditional FIMs proposed by Wen et al. [6] and Li et al. [5], the FIM using Chebyshev polynomials and their analytical solutions.

2 FIM by Using Legendre Polynomials

In this section, we construct the FIM by modifiying the idea of Duangpan [3] to construct the first order finite integration matrix base on the Legendre polynomial expansion. Then, the m^{th} order finite integration matrix can be obtained easily. Now, let us introduce the Legendre polynomial and some useful facts about it.

Definition 2.1. ([1]) For $x \in [-1, 1]$, the Legendre polynomial of degree $n \geq 0$

is recursively defined as

$$(n+1)L_{n+1}(x) - (2n+1)xL_n(x) + nL_{n-1}(x) = 0, \text{for } n \geq 1, \tag{2.1}$$

where $L_0(x) = 1$ and $L_1(x) = x$.

The following properties of the Legendre polynomials $L_n(x)$ help us construct the first and the higher order integration matrices as well as the procedure for our FIM.

Lemma 2.2. *(i) the Legendre polynomial of degree n has n distinct roots in the interval $(-1,1)$.*
(ii) For $x \in [-1,1]$,

$$\bar{L}_0(x) := \int_{-1}^{x} L_0(\xi)d\xi = x+1 \textit{ and} \tag{2.2}$$

$$\bar{L}_n(x) := \int_{-1}^{x} L_n(\xi)d\xi = \frac{1}{2n+1}(L_{n+1}(x) - L_{n-1}(x)) \textit{ for } n \geq 1. \tag{2.3}$$

(iii) For a nonnegative integer N, the discrete orthogonality relation of Legendre polynomial is

$$\sum_{k=0}^{N} L_i(\bar{x}_k)L_j(\bar{x}_k) = \begin{cases} 0 & \textit{if } i \neq j \\ N+1 & \textit{if } i = j = 0 \\ \frac{2}{2N+1} & \textit{if } i = j \neq 0 \end{cases}, \tag{2.4}$$

where $\bar{x}_k, k \in \{0,1,2,...,N\}$, are zeros of $L_{n+1}(x)$, and $0 \leq i,j \leq N$.

Proof. (i) and (iii) See [1].

(ii) Let $x \in [-1,1]$. We obtain easily that

$$\bar{L}_0(x) = \int_{-1}^{x} L_0(\xi)d\xi = \int_{-1}^{x} 1 d\xi = x+1.$$

Next, let $n \geq 1$ and $S_{n+1}(x) = \int_{-1}^{x} L_n(\xi)d\xi$. Hence, S_{n+1} is a polynomial of degree $n+1$ and $S_n(\pm 1) = 0$. Therefore, for any $m < n-1$, we can use integration by parts to obtain

$$\int_{-1}^{1} S_{n+1}L_m dx = \int_{-1}^{1} S_{n+1}S'_{m+1}dx = -\int_{-1}^{1} S'_{n+1}S_{m+1}dx = \int_{-1}^{1} L_n S_{m+1}dx = 0.$$

Hence, we can write $S_{n+1} = a_{n-1}L_{n-1} + a_nL_n + a_{n+1}L_{n+1}$. By parity argument, $a_n = 0$.

On the other hand, by writing $L_n = k_n x^n + k_{n-1}x^{n-1} + \cdots + k_0$, we find from the definition of S_{n+1} that $\frac{k_n}{n+1} = a_{n+1}k_{n+1}$. We then derive from the formula of k_n that $a_{n+1} = \frac{1}{2n+1}$. Finally, we derive from $S_{n+1}(-1) = 0$ that $a_{n-1} = -a_{n+1} = -\frac{1}{2n+1}$. □

Next, for a nonnegative integer N, let the Legendre matrix $\mathbf{L}$ be defined as

$$\mathbf{L} = \begin{bmatrix} L_0(\bar{x}_0) & L_1(\bar{x}_0) & \cdots & L_N(\bar{x}_0) \\ L_0(\bar{x}_1) & L_1(\bar{x}_1) & \cdots & L_N(\bar{x}_1) \\ \vdots & \vdots & \ddots & \vdots \\ L_0(\bar{x}_N) & L_1(\bar{x}_N) & \cdots & L_N(\bar{x}_N) \end{bmatrix}.$$

That is, $\mathbf{L}$ is the matrix whose elements are Legendre polynomials evaluated at the zeros $\bar{x}_k$ of the Legendre polynomial $L_{N+1}(x)$ for $k \in \{0, 1, 2, ..., N\}$.

Lemma 2.3. *$\boldsymbol{L}$ has an inverse which is*

$$\boldsymbol{L}^{-1} = \frac{1}{N+1}\begin{bmatrix} \frac{L_0(\bar{x}_0)}{N+1} & L_0(\bar{x}_1) & \cdots & L_0(\bar{x}_N) \\ L_1(\bar{x}_0) & \frac{2L_1(\bar{x}_1)}{2N+1} & \cdots & L_1(\bar{x}_N) \\ \vdots & \vdots & \ddots & \vdots \\ L_N(\bar{x}_0) & L_N(\bar{x}_1) & \cdots & \frac{2L_N(\bar{x}_N)}{2N+1} \end{bmatrix}.$$

Proof. It comes directly from Lemma 2.2 (iii). □

Let N be a nonnegative integer and the approximate solution $u(x)$ be a linear combination of the Legendre polynomials $L_0(x), L_1(x), L_2(x), ..., L_N(x)$. That is,

$$u(x) = \sum_{n=0}^{N} c_n L_n(x), \text{ for } x \in [-1, 1]. \tag{2.5}$$

Let $-1 \le \bar{x}_0 < \bar{x}_1 < \bar{x}_2 < ... < \bar{x}_N \le 1$ be grid points that is generated by the zeros of Legendre polynomial $L_{N+1}(x)$ distributed on $[-1, 1]$. Then, by (2.5), we have

$$u(\bar{x}_k) = \sum_{n=0}^{N} c_n L_n(\bar{x}_k)$$

for $k \in \{0, 1, 2, ..., N\}$ or,

$$\begin{bmatrix} u(\bar{x}_0) \\ u(\bar{x}_1) \\ \vdots \\ u(\bar{x}_N) \end{bmatrix} = \begin{bmatrix} L_0(\bar{x}_0) & L_1(\bar{x}_0) & \cdots & L_N(\bar{x}_0) \\ L_0(\bar{x}_1) & L_1(\bar{x}_1) & \cdots & L_N(\bar{x}_1) \\ \vdots & \vdots & \ddots & \vdots \\ L_0(\bar{x}_N) & L_1(\bar{x}_N) & \cdots & L_N(\bar{x}_N) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix},$$

which is denoted by $\mathbf{u} = \mathbf{Lc}$. Note that, each element of $\mathbf{L}$ can be found by using (2.1). Thus, the coefficients $\{c_n\}_{n=0}^{N}$ can be defined by $\mathbf{c} = \mathbf{L}^{-1}\mathbf{u}$. Next, for $k \in \{0,1,2,...,N\}$, let us consider the single integral of $u(x)$ from -1 to $\bar{x}_k$, which is denoted by $U(\bar{x}_k)$. Then,

$$U(\bar{x}_k) = \int_{-1}^{\bar{x}_k} u(\xi)d\xi = \sum_{n=0}^{N} c_n \int_{-1}^{\bar{x}_k} L_n(\xi)d\xi = \sum_{n=0}^{N} c_n \bar{L}_n(\bar{x}_k)$$

for $k \in \{0,1,2,...,N\}$ or,

$$\begin{bmatrix} U(\bar{x}_0) \\ U(\bar{x}_1) \\ \vdots \\ U(\bar{x}_N) \end{bmatrix} = \begin{bmatrix} \bar{L}_0(\bar{x}_0) & \bar{L}_1(\bar{x}_0) & \cdots & \bar{L}_N(\bar{x}_0) \\ \bar{L}_0(\bar{x}_1) & \bar{L}_1(\bar{x}_1) & \cdots & \bar{L}_N(\bar{x}_1) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{L}_0(\bar{x}_N) & \bar{L}_1(\bar{x}_N) & \cdots & \bar{L}_N(\bar{x}_N) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix},$$

which is denoted by $\mathbf{U} = \bar{\mathbf{L}}\mathbf{c} = \bar{\mathbf{L}}\mathbf{L}^{-1}\mathbf{u}$. Note that, each element of $\bar{\mathbf{L}}$ can be found by using (2.2) and (2.3). Next, by letting $\mathbf{A} = \bar{\mathbf{L}}\mathbf{L}^{-1}$, we have $\mathbf{U} = \mathbf{Au}$. This $\mathbf{A} = [a_{ki}]_{(N+1)\times(N+1)}$ is called *the first order integration matrix* for FIM by using Legendre polynomials, i.e.,

$$U(\bar{x}_k) = \int_{-1}^{\bar{x}_k} u(\xi)d\xi = \sum_{i=0}^{N} a_{ki} u(\bar{x}_i)$$

for $k \in \{0,1,2,...,N\}$ or,

$$\begin{bmatrix} U(\bar{x}_0) \\ U(\bar{x}_1) \\ \vdots \\ U(\bar{x}_N) \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0N} \\ a_{10} & a_{11} & \cdots & a_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N0} & a_{N1} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} u(\bar{x}_0) \\ u(\bar{x}_1) \\ \vdots \\ u(\bar{x}_N) \end{bmatrix}.$$

Now, for $k \in \{0,1,2,...,N\}$, let us consider the double integral of $u(x)$ from -1 to $\bar{x}_k$, which is denoted by $U^{(2)}(\bar{x}_k)$. Then,

$$\begin{aligned} U^{(2)}(\bar{x}_k) &= \int_{-1}^{\bar{x}_k} \int_{-1}^{\xi_2} u(\xi_1)d\xi_1 d\xi_2 \\ &= \sum_{i=0}^{N} a_{ki} \int_{-1}^{\bar{x}_i} u(\xi_1)d\xi_1 = \sum_{i=0}^{N}\sum_{j=0}^{N} a_{ki}a_{ij}u(\bar{x}_j) = \sum_{i=0}^{N} a_{ki}^{(2)} u(\bar{x}_i) \end{aligned}$$

in which we can write in the matrix form as $\mathbf{U}^{(2)} = \mathbf{A}^{(2)}\mathbf{u}$. Since the summation $\sum_{j=0}^{N} a_{ki}a_{ij}$ represents each element in A^2, we can conclude that $\mathbf{U}^{(2)} = \mathbf{A}^{(2)}\mathbf{u} =$

$\mathbf{A}^2\mathbf{u}$. Similarly, for $k \in \{0,1,2,...,N\}$, we can construct the multi-layer integral of $u(x)$ from -1 to $\bar{x}_k$ by using the same idea as to construct $\mathbf{A}^{(2)}$. That is,

$$\begin{aligned} U^{(m)}(\bar{x}_k) &= \int_{-1}^{\bar{x}_k} \cdots \int_{-1}^{\xi_2} u(\xi_1)d\xi_1 \cdots d\xi_m \\ &= \sum_{i_m=1}^{N} \cdots \sum_{j=1}^{N} a_{ki_m} \cdots a_{i_1 j} u(\bar{x}_j) = \sum_{i=1}^{N} a_{ki}^{(m)} u(\bar{x}_i) \end{aligned}$$

whose matrix form can be expressed as $\mathbf{U}^{(m)} = \mathbf{A}^{(m)}\mathbf{u} = \mathbf{A}^m\mathbf{u}$. This $\mathbf{A}^{(m)}$ is called *the m^{th} order integration matrix.*

For boundary conditions, we may need higher order derivatives which can be achieved by considering derivatives of the linear combination of Legendre polynomials at the end point $x = \pm 1$ using (2.3). Thus, we have

$$u(\pm 1) = \sum_{n=0}^{N} c_n L_n(\pm 1) \quad \text{and} \quad u^{(p)}(\pm 1) = \sum_{n=0}^{N} c_n \frac{d^p}{dx^p} L_n(x)\Big|_{x=\pm 1}$$

for $p \in \mathbb{N}$.

3 Procedure for Solving Linear ODEs

In this section, we devise our proposed modified FIM to construct an algorithm for solving linear ODEs with boundary conditions. Usually, the linear ODE is given by $Pu(x) = f(x)$ for $x \in (a,b)$, where $P = a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + a_{n-2}(x)\frac{d^{n-2}}{dx^{n-2}} + ... + a_0(x)$. The procedure is given as follows,

Step 1. We transform $x \in [a,b]$ into $\bar{x} \in [-1,1]$ by the transformation $\bar{x} = \frac{2x-a-b}{b-a}$. Let $k = \frac{2}{b-a}$. Then, $Pu(x) = f(x)$ for $x \in (a,b)$ becomes

$$\hat{P}u(\bar{x}) = f(\bar{x}) \text{ for } \bar{x} \in (-1,1), \tag{3.1}$$

where $\hat{P} = k^n a_n(\bar{x})\frac{d^n}{d\bar{x}^n} + k^{n-1}a_{n-1}(\bar{x})\frac{d^{n-1}}{d\bar{x}^{n-1}} + k^{n-2}a_{n-2}(\bar{x})\frac{d^{n-2}}{d\bar{x}^{n-2}} + ... + a_0(\bar{x})$. Thus, after this step, we will consider the problem in [-1,1].

Step 2. We discretize our domain $[-1,1]$ into N subintervals with $N+1$ nodes. The grid points is generated by the zeros of Legendre polynomial $L_{N+1}(x)$.

Step 3. We eliminate derivatives out of (3.1) by taking n layers integral from -1 to $\bar{x}_k$ on both sides of (3.1) and using integration by parts. Thus, we obtain

$$k^n a_n(\bar{x}_k)u(\bar{x}_k)+k^{n-1}a_{n-1}(\bar{x}_k)\int_{-1}^{\bar{x}_k} u(\xi_1)d\xi_1+k^{n-2}a_{n-2}(\bar{x}_k)\int_{-1}^{\bar{x}_k}\int_{-1}^{\xi_2} u(\xi_1)d\xi_1 d\xi_2$$

$$+k^{n-3}a_{n-3}(\bar{x}_k)\int_{-1}^{\bar{x}_k}\int_{-1}^{\xi_3}\int_{-1}^{\xi_2} u(\xi_1)d\xi_1 d\xi_2 d\xi_3+...+a_0(\bar{x}_k)\int_{-1}^{\bar{x}_k}...\int_{-1}^{\xi_2} u(\xi_1)d\xi_1...d\xi_n$$

$$=\int_{-1}^{\bar{x}_k}...\int_{-1}^{\xi_2} f(\xi_1)d\xi_1...d\xi_n + d_1\frac{\bar{x}_k^{n-1}}{(n-1)!} + d_2\frac{\bar{x}_k^{n-2}}{(n-2)!} + d_3\frac{\bar{x}_k^{n-3}}{(n-3)!} + ... + d_n,$$

where $d_1, d_2, d_3, ..., d_n$ are the arbitrary integral constants.

Step 4. By the idea described in Section 2, we write $u(x)$ as a linear combination of the Legendre polynomials and by using the m^{th} order integration matrix developed in Section 2, we can transform the equation in Step 3 in the matrix form as follow,

$$\begin{aligned} &k^n\mathbf{B}_n\mathbf{u} + k^{n-1}\mathbf{A}\mathbf{B}_{n-1}\mathbf{u} + k^{n-2}\mathbf{A}^2\mathbf{B}_{n-2}\mathbf{u} + ... + \mathbf{A}^n\mathbf{B}_0\mathbf{u} \\ &\qquad = \mathbf{A}^n\mathbf{f} + d_1\mathbf{x}_{n-1} + d_2\mathbf{x}_{n-2} + ... + d_n\mathbf{i}. \end{aligned}$$

Let $\mathbf{K} = k^n\mathbf{B}_n + k^{n-1}\mathbf{A}\mathbf{B}_{n-1} + k^{n-2}\mathbf{A}^2\mathbf{B}_{n-2} + ... + \mathbf{A}^n\mathbf{B}_0$. Then, we have

$$\mathbf{K}\mathbf{u} = \mathbf{A}^n\mathbf{f} + d_1\mathbf{x}_{n-1} + d_2\mathbf{x}_{n-2} + ... + d_n\mathbf{i}, \tag{3.2}$$

where

$$\begin{aligned} \mathbf{u} &= [u(\bar{x}_0), u(\bar{x}_1), u(\bar{x}_2), ..., u(\bar{x}_N)]^T, \\ \mathbf{f} &= [f(\bar{x}_0), f(\bar{x}_1), f(\bar{x}_2), ..., f(\bar{x}_N)]^T, \\ \mathbf{i} &= [1, 1, 1, ..., 1]_{N+1}^T, \\ \mathbf{x}_i &= \frac{1}{i!}[\bar{x}_0^i, \bar{x}_1^i, \bar{x}_2^i, ..., \bar{x}_N^i]^T \text{ for } i \in \{1, 2, 3, ..., n-1\}, \\ \mathbf{B}_i &= \text{diag}[a_i(\bar{x}_0), a_i(\bar{x}_1), a_i(\bar{x}_2), ..., a_i(\bar{x}_N)] \text{ for } i \in \{0, 1, 2, ..., n\} \text{ and} \\ \mathbf{A} &= \bar{\mathbf{L}}\mathbf{L}^{-1} \end{aligned}$$

as defined in Section 2.

Step 5. We consider the boundary conditions and change it into the vector form. For $p \in \mathbb{N}$, we get

$$u(\pm 1) = \sum_{n=0}^{N} c_n (\pm 1)^n = \mathbf{t}_0 \mathbf{c} = \mathbf{t}_0 \mathbf{L}^{-1} \mathbf{u} \text{ and}$$
$$u^{(p)}(\pm 1) = \sum_{n=0}^{N} c_n \frac{d^p}{dx^p} L_n(x)\Big|_{x=\pm 1} = \mathbf{t}_p \mathbf{c} = \mathbf{t}_p \mathbf{L}^{-1} \mathbf{u}.$$

where $t_0 = [(\pm 1)^0 \ (\pm 1)^1 \ (\pm 1)^2 \ \ldots \ (\pm 1)^N]^T$ and $t_p = [L_0^{(p)}(\pm 1)^0 \ L_0^{(p)}(\pm 1)^1 \ L_0^{(p)}(\pm 1)^2 \ \ldots \ L_0^{(p)}(\pm 1)^N]^T$.

Step 6. From (3.2) in Step 4, we can rearrange it to obtain the linear system as

$$\mathbf{K}\mathbf{u} - d_1 \mathbf{x}_{n-1} - d_2 \mathbf{x}_{n-2} - \ldots - d_n \mathbf{i} = \mathbf{A}^n \mathbf{f}.$$

Together with the boundary conditions, for example, $u'(\pm 1) = b_1$, $u''(\pm 1) = b_2$, ..., $u^{(n)}(\pm 1) = b_n$ which can be represented in the vector form as

$$\begin{aligned} u'(\pm 1) &= \mathbf{t}_1 \mathbf{L}^{-1} \mathbf{u} = b_1 \\ u''(\pm 1) &= \mathbf{t}_2 \mathbf{L}^{-1} \mathbf{u} = b_2 \\ &\vdots \\ u^{(n)}(\pm 1) &= \mathbf{t}_n \mathbf{L}^{-1} \mathbf{u} = b_n. \end{aligned}$$

Step 7. Finally, we obtain the linear system in a matrix form as follows

$$\left[\begin{array}{c|cccc} \mathbf{K} & -\mathbf{x}_{n-1} & -\mathbf{x}_{n-2} & \cdots & -\mathbf{i} \\ \hline \mathbf{t}_1 \mathbf{T}^{-1} & 0 & 0 & \cdots & 0 \\ \mathbf{t}_2 \mathbf{T}^{-1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{t}_n \mathbf{T}^{-1} & 0 & 0 & \cdots & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{u} \\ \hline d_1 \\ d_2 \\ \vdots \\ d_n \end{array}\right] = \left[\begin{array}{c} \mathbf{A}^n \mathbf{f} \\ \hline b_1 \\ b_2 \\ \vdots \\ b_n \end{array}\right]. \tag{3.3}$$

By solving the linear system (3.3), we will obtain $u(\bar{x})$ for $\bar{x} \in [-1, 1]$. Then, by using the transformation $x = \frac{1}{2}[(b-a)\bar{x} + a + b]$, we can finally obtain the approximate solution $u(x)$ for $x \in [a, b]$.

Remark 3.1. If we discretize our domain $[-1, 1]$ into N subintervals with $N+1$ nodes, the coefficient matrix in (3.3) is of dimension $N + n + 1$, where n is the order of the ODE.

4 Numerical Examples

In this section, we apply our modified FIM by using Chebyshev polynomials to finding approximation solution of linear ordinary differential equations as shown in Examples 4.1-4.4 to compare with the FDM, the FIM with trapezoidal rule and the FIM with Chebyshev's polynomials. We use the average relative error to demonstrate the accuracy between these methods and their analytical solutions, as shown in Tables 1-4. The average relative error is computed by $\frac{1}{N+1}\sum_{i=0}^{N}\frac{|u(x_i)-u^*(x_i)|}{|u^*(x_i)|}$.

Example 4.1. We consider the following boundary value problem for ordinary differential equation

$$\frac{d^2u}{dx^2} - u = 0 \text{ for } x \in (0,1)$$

with boundary conditions $u(0) = 0$ and $u(1) = 1$. The analytical solution is $u^*(x) = \frac{e^x - e^{-x}}{e - e^{-1}}$.

Transform $x \in [a,b] = [0,1]$ into $\bar{x} \in [-1,1]$ by using $\bar{x} = 2x - 1$. Let $k = \frac{2}{b-a} = 2$, we get $k^2 u''(\bar{x}) - u(\bar{x}) = 0$. Applying integration operation twice on both sides of the equation, we have $k^2\mathbf{u} - \mathbf{A}^2\mathbf{u} = d_1\mathbf{x} + d_2\mathbf{i}$. From boundary conditions, $u(0) = \mathbf{t}_{0l}\mathbf{L}^{-1}\mathbf{u} = 0$ and $u(1) = \mathbf{t}_{0r}\mathbf{L}^{-1}\mathbf{u} = 1$. Thus, we can construct the linear system (3.3) in the matrix form as

$$\left[\begin{array}{c|cc} k^2\mathbf{I} - \mathbf{A}^2 & -\mathbf{x} & -\mathbf{i} \\ \hline \mathbf{t}_{0l}\mathbf{L}^{-1} & 0 & 0 \\ \mathbf{t}_{0r}\mathbf{L}^{-1} & 0 & 0 \end{array}\right]\left[\begin{array}{c} \mathbf{u} \\ \hline d_1 \\ d_2 \end{array}\right] = \left[\begin{array}{c} \mathbf{0} \\ \hline 0 \\ 1 \end{array}\right].$$

Example 4.2. We consider the following initial value problem for ordinary differential equation

$$\frac{d^2u}{dx^2} - 2(1+2x^2)u = 0 \text{ for } x \in (0,1)$$

with initial conditions $u(0) = 1$ and $u'(0) = 0$. The analytical solution is $u^*(x) = e^{x^2}$.

Transform $x \in [a,b] = [0,1]$ into $\bar{x} \in [-1,1]$ by using $\bar{x} = 2x - 1$. Let $k = \frac{2}{b-a} = 2$, we get $k^2 u''(\bar{x}) - 2(1 + 2(\frac{\bar{x}+1}{2})^2)u(\bar{x}) = 0$. Taking twice integration on both sides of the equation, we have $k^2\mathbf{u} - \mathbf{A}^2\mathbf{B}\mathbf{u} = d_1\mathbf{x} + d_2\mathbf{i}$ where $\mathbf{B} = \mathrm{diag}[2(1 + 2(\frac{\bar{x}_0+1}{2})^2), 2(1 + 2(\frac{\bar{x}_1+1}{2})^2), 2(1 + 2(\frac{\bar{x}_2+1}{2})^2), ..., 2(1 + 2(\frac{\bar{x}_N+1}{2})^2)]$. By initial conditions, $u(0) = \mathbf{t}_{0l}\mathbf{L}^{-1}\mathbf{u} = 1$ and $u'(0) = \mathbf{t}_{1l}\mathbf{L}^{-1}\mathbf{u} = 0$. Thus, we can construct the linear system (3.3) in the matrix form as

$$\left[\begin{array}{c|cc} k^2\mathbf{I}-\mathbf{A}^2\mathbf{B} & -\mathbf{x} & -\mathbf{i} \\ \hline \mathbf{t}_{0l}\mathbf{L}^{-1} & 0 & 0 \\ \mathbf{t}_{1l}\mathbf{L}^{-1} & 0 & 0 \end{array}\right]\left[\begin{array}{c} \mathbf{u} \\ \hline d_1 \\ d_2 \end{array}\right]=\left[\begin{array}{c} \mathbf{0} \\ \hline 1 \\ 0 \end{array}\right].$$

Example 4.3. We consider the following boundary value problem for ordinary differential equation

$$\frac{d^2u}{dx^2}+2\frac{du}{dx}+u=\sin x \text{ for } x\in(0,2)$$

with $u(0)=0$ and $u(2)=-1$. The analytical solution is $u^*(x)=\frac{e^{-x}}{4}(xe^2\cos 2-2e^x\cos x-2xe^2-x+2)$.

Transform $x\in[a,b]=[0,2]$ into $\bar{x}\in[-1,1]$ by using $\bar{x}=x-1$. Let $k=\frac{2}{b-a}=1$, we get $k^2u''(\bar{x})+2ku'(\bar{x})+u(\bar{x})=\sin\bar{x}$. Applying double integral on both sides of the equation, we have $k^2\mathbf{u}-2k\mathbf{A}\mathbf{u}+\mathbf{A}^2\mathbf{u}=\mathbf{A}^2\mathbf{f}+d_1\mathbf{x}+d_2\mathbf{i}$ where $\mathbf{f}=[\sin(\bar{x}_0+1)\quad \sin(\bar{x}_1+1)\quad \sin(\bar{x}_2+1)\quad \ldots\quad \sin(\bar{x}_N+1)]^T$. From boundary conditions, $u(0)=\mathbf{t}_{0l}\mathbf{L}^{-1}\mathbf{u}=0$ and $u(2)=\mathbf{t}_{0r}\mathbf{L}^{-1}\mathbf{u}=-1$. Thus, we can construct the linear system (3.3) in the matrix form as

$$\left[\begin{array}{c|cc} k^2\mathbf{I}+2k\mathbf{A}+\mathbf{A}^2 & -\mathbf{x} & -\mathbf{i} \\ \hline \mathbf{t}_{0l}\mathbf{L}^{-1} & 0 & 0 \\ \mathbf{t}_{0r}\mathbf{L}^{-1} & 0 & 0 \end{array}\right]\left[\begin{array}{c} \mathbf{u} \\ \hline d_1 \\ d_2 \end{array}\right]=\left[\begin{array}{c} \mathbf{A}^2\mathbf{f} \\ \hline 0 \\ -1 \end{array}\right].$$

Example 4.4. We consider the following boundary value problem for ordinary differential equation

$$\frac{d^4u}{dx^4}+u=1 \text{ for } x\in(0,1)$$

with boundary conditions $u(0)=u(1)=u''(0)=u''(1)=0$. The analytical solution is obtained as

$$u^*(x)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi x}{(2n-1)\big[(2n-1)^4\pi^4+1\big]}.$$

Transform $x\in[a,b]=[0,1]$ into $\bar{x}\in[-1,1]$ by using $\bar{x}=2x-1$. Let $k=\frac{2}{b-a}=2$, we get $k^4u^{(4)}(\bar{x})+u(\bar{x})=1$. Taking four-layer integration on both sides of the equation, we have $k^4\mathbf{u}+\mathbf{A}^4\mathbf{u}=\mathbf{A}^4\mathbf{i}+d_1\frac{\mathbf{x}^3}{6}+d_2\frac{\mathbf{x}^2}{2}+d_3\mathbf{x}+d_4\mathbf{i}$. From boundary conditions, $u(0)=\mathbf{t}_{0l}\mathbf{L}^{-1}\mathbf{u}=0$, $u(1)=\mathbf{t}_{0r}\mathbf{L}^{-1}\mathbf{u}=0$, $u''(0)=\mathbf{t}_{2l}\mathbf{L}^{-1}\mathbf{u}=0$ and $u''(1)=\mathbf{t}_{2r}\mathbf{L}^{-1}\mathbf{u}=0$. Thus, we can construct the linear system

(3.3) in the matrix form as

$$\left[\begin{array}{c|cccc} k^4\mathbf{I}+\mathbf{A}^4 & -\frac{\mathbf{x}^3}{6} & -\frac{\mathbf{x}^2}{2} & -\mathbf{x} & -\mathbf{i} \\ \hline \mathbf{t}_{0l}\mathbf{L}^{-1} & 0 & 0 & 0 & 0 \\ \mathbf{t}_{0r}\mathbf{L}^{-1} & 0 & 0 & 0 & 0 \\ \mathbf{t}_{2l}\mathbf{L}^{-1} & 0 & 0 & 0 & 0 \\ \mathbf{t}_{2r}\mathbf{L}^{-1} & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} \mathbf{u} \\ \hline d_1 \\ d_2 \\ d_3 \\ d_4 \end{array}\right] = \left[\begin{array}{c} \mathbf{A}^4\mathbf{i} \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right].$$

Table 1 : Average relative error for Example 4.1

N	FDM	FIM(TPZ)	FIM(CBS)	FIM(LEG)
4	4.2348×10^{-4}	8.6307×10^{-4}	1.2734×10^{-6}	1.4237×10^{-3}
6	2.0604×10^{-4}	4.1554×10^{-4}	2.8757×10^{-8}	5.2064×10^{-6}
8	1.2070×10^{-4}	2.4254×10^{-4}	3.5836×10^{-11}	8.7122×10^{-9}
10	7.9045×10^{-5}	1.5857×10^{-4}	2.0366×10^{-14}	8.3104×10^{-12}
12	5.5711×10^{-5}	1.1165×10^{-4}	1.3307×10^{-15}	1.3057×10^{-15}

Table 2 : Average relative error for Example 4.2

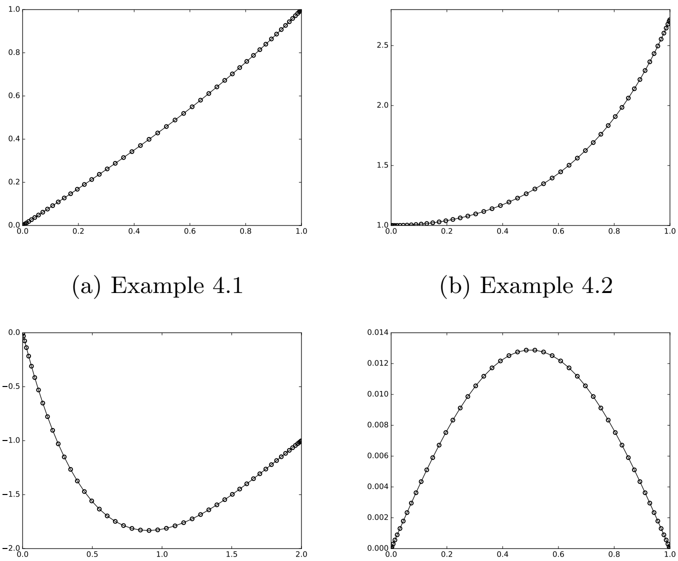
N	FDM	FIM(TPZ)	FIM(CBS)	FIM(LEG)
4	9.8768×10^{-2}	2.6416×10^{-2}	3.6456×10^{-2}	1.0414×10^{-1}
6	6.5513×10^{-2}	1.0750×10^{-2}	1.6877×10^{-3}	1.0016×10^{-2}
8	4.8888×10^{-2}	5.8304×10^{-3}	5.0529×10^{-5}	4.1923×10^{-4}
10	3.8964×10^{-2}	3.6583×10^{-3}	1.0802×10^{-6}	1.1394×10^{-5}
12	3.2379×10^{-2}	2.5093×10^{-3}	1.7839×10^{-8}	2.2826×10^{-7}

Table 3 : Average relative error for Example 4.3

N	FDM	FIM(TPZ)	FIM(CBS)	FIM(LEG)
4	5.6121×10^{-2}	9.9239×10^{-3}	2.0626×10^{-3}	6.3162×10^{-2}
6	2.5671×10^{-2}	4.7917×10^{-3}	1.9124×10^{-5}	1.2867×10^{-3}
8	1.4780×10^{-2}	2.8101×10^{-3}	8.9593×10^{-8}	9.7721×10^{-6}
10	9.6178×10^{-3}	1.8443×10^{-3}	2.5238×10^{-10}	4.4627×10^{-8}
12	6.7606×10^{-3}	1.3026×10^{-3}	4.3821×10^{-13}	1.2049×10^{-10}

Table 4 : Average relative error for Example 4.4

N	FDM	FIM(TPZ)	FIM(CBS)	FIM(LEG)
4	5.4712×10^{-3}	6.3349×10^{-4}	5.4978×10^{-6}	6.9862×10^{-3}
6	4.6006×10^{-3}	3.1233×10^{-4}	1.6147×10^{-7}	9.3062×10^{-6}
8	3.8372×10^{-3}	1.8436×10^{-4}	2.8532×10^{-11}	3.1485×10^{-7}
10	3.2629×10^{-3}	1.2133×10^{-4}	2.0896×10^{-13}	6.7363×10^{-11}
12	2.8290×10^{-3}	8.5804×10^{-5}	1.5383×10^{-14}	5.9514×10^{-13}



(a) Example 4.1 (b) Example 4.2

(c) Example 4.3 (d) Example 4.4

Figure 1: The graphs of solution in Examples 4.1-4.4 for $N = 50$.

5 Conclusion and Discussion

In this paper, we modify the FIM by using Legendre polynomial for finding approximate solutions to linear ODE. Our modified FIM has much higher accuracy than the FDM and the traditional FIMs such as trapezoidal rule as shown in the several examples in Section 4. However, it is about the same accuracy as Duangpan obtained by using FIM with Chebyshev polynomial. One may worry about the computational time. Since our modified FIM using a larger computational matrix compared to the FDM, in the case of using the same number of nodes. However, for the same accuracy, one can see in Table 5 that the FDM needs a lot more nodes, and hence, a lot larger matrix dimension to solve the systems of linear equations involved comparing to our modified FIM.

Table 5 : Dimension of matrix involved when considering the same accuracy

Example	Average Relative Error	Dimension of Matrix Involved	
		FIM(LEG)	FDM
4.1	5.2064×10^{-6}	9	83
4.2	3.9881×10^{-2}	8	12
4.3	1.2867×10^{-3}	9	24
4.4	9.3062×10^{-6}	11	> 100

Our future work is try to apply our modified FIM for solving boundary value problems to partial or nonlinear differential equations.

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