Chamchuri Journal of Mathematics
Volume 1(2009) Number 1, 81–86
http://www.math.sc.chula.ac.th/cjm

A Non-regular Continued Fraction and Its Characterization Property

Vichian Laohakosol and Panupong Vichitkunakorn*

Received 26 June 2008
Revised 5 May 2009
Accepted 6 May 2009

Abstract: A construction of non-regular continued fractions is introduced. Some basic properties and its characterization property are derived.

Keywords: non-regular continued fraction

2000 Mathematics Subject Classification: 11A55, 11F70

1 Introduction

In the field of formal series $\mathbb{F}((x^{-1}))$ equipped with the degree valuation, an expression of the form:

$$b_0 + \frac{a_1}{b_1+}\ \frac{a_2}{b_2+}\ \frac{a_3}{b_3+} \ldots := [b_0;\ a_1,\ b_1;\ a_2,\ b_2;\ a_3,\ b_3; \ldots], \tag{1}$$

where $a_i,\ b_i \in \mathbb{F}[x]$, is called a (non-regular) continued fraction. The elements $a_i,\ b_i$ are called *partial numerators*, and *partial denominators*, respectively. When all $a_i = 1$, and all $b_i \in \mathbb{F}[x] \setminus \mathbb{F}$ $(i \geq 1)$, we usually write $[b_0,\ b_1,\ b_2, \ldots]$ for $[b_0;\ 1, b_1;\ 1, b_2;\ 1, \ldots]$ and call it a *regular continued fractions.* A finite continued fraction $[b_0; a_1, b_1; a_2, b_2; \ldots ; a_n, b_n]$ is called the nth convergent of the continued fraction (1). It is well-known, see e.g. [1], that every $\xi \in \mathbb{F}((x^{-1}))$ can be uniquely represented by a regular continued fractions, i.e.,

$$\xi = [b_0; b_1, b_2, b_3, \ldots] = b_0 + \frac{1}{b_1+}\ \frac{1}{b_2+}\ \frac{1}{b_3+} \ldots .$$

* *Corresponding author*

In this work, we devise an algorithm to construct a non-regular continued fraction, with prescribed partial numerators, for each element in $\mathbb{F}((x^{-1}))$ uniquely. If the prescribed sequence of partial numerators is $\{a_n\} = \{1, 1, 1, \ldots\}$, then we recover its regular continued fraction expansion. Apart from proposing the algorithm, its basic properties and its use of rationality characterization are also established.

2 Construction

Throughout, $\mathbf{F}$ denotes $\mathbb{F}((x^{-1}))$, the field of formal series over a field $\mathbb{F}$. Let $\{a_n\}_{n=1}^{\infty}$ be a fixed sequence of *partial numerators* in $\mathbb{F}[x]\backslash\{0\}$. Since elements of $\mathbf{F}$ are formal series (in x) written uniquely under the form:

$$\xi = c_m x^m + c_{m-1}x^{m-1} + c_{m-2}x^{m-2} + \ldots,$$

where the coefficients $c_m, c_{m-1}, c_{m-2}, \ldots$ are in $\mathbb{F}$. The degree valuation $|\cdot|$, which is non-archimedean, in $\mathbf{F}$ is defined by

$$|0| = 0, \quad |\xi| = 2^m \text{ if } \xi = c_m x^m + c_{m-1}x^{m-1} + c_{m-2}x^{m-2} + \ldots, \; c_m \neq 0.$$

The construction of our non-regular continued fraction for ξ with respect to a given sequence of partial numerators $\{a_i\} \subseteq F[x]$, *each of whose elements is assumed nonzero*, runs as follows: We define

$$\xi = [\xi] + (\xi),$$

where

$$\begin{aligned} [\xi] &:= c_m x^m + c_{m-1}x^{m-1} + c_{m-2}x^{m-2} + \ldots + c_1 x + c_0, \\ (\xi) &:= c_{-1}x^{-1} + c_{-2}x^{-2} + \ldots. \end{aligned}$$

are referred to as the *head* and the *tail* parts of ξ, respectively. Clearly, the head and the tail parts of ξ are uniquely determined. Let

$$b_0 = [\xi] \in \mathbb{F}[x].$$

Then $|b_0| = |\xi| \geq 1$, provided $[\xi] \neq 0$. If $(\xi) = 0$, then the process stops. If $(\xi) \neq 0$, then write $\xi = b_0 + \frac{1}{\beta_1}$, where $\beta_1 = \frac{1}{(\xi)}$ and so $|\beta_1| > 1$. Now write

$$\xi = b_0 + \frac{a_1}{a_1\beta_1}.$$

Since $1 \le |a_1| < |a_1\beta_1|$, we can write $a_1\beta_1 = [a_1\beta_1] + (a_1\beta_1)$. Let

$$b_1 = [a_1\beta_1] \in \mathbb{F}[x] \setminus \mathbb{F},$$

so that $|b_1| = |a_1\beta_1| > |a_1|$. If $(a_1\beta_1) = 0$, then the process stops. If $(a_1\beta_1) \neq 0$, then write $a_1\beta_1 = b_1 + \frac{1}{\beta_2}$, where $\beta_2 = \frac{1}{(a_1\beta_1)}$ so that $|\beta_2| > 1$. Next write

$$a_1\beta_1 = b_1 + \frac{a_2}{a_2\beta_2}.$$

Since $1 \le |a_2| < |a_2\beta_2|$, put

$$b_2 = [a_2\beta_2] \in \mathbb{F}[x] \setminus \mathbb{F},$$

so that $|b_2| = |a_2\beta_2| > |a_2|$. Again, if $(a_2\beta_2) = 0$, then the process stops. If $(a_2\beta_2) \neq 0$, then continue in the same manner. By so doing, we obtain a representation

$$\begin{aligned} \xi &= [b_0 : \ a_1,\ b_1;\ a_2,\ b_2; \ldots;\ a_{n-1},\ b_{n-1};\ a_n,\ a_n\beta_n] \\ &:= b_0 + \frac{a_1}{b_1+}\ \frac{a_2}{b_2+} \cdots \ \frac{a_{n-1}}{b_{n-1}+}\ \frac{a_n}{a_n\beta_n}, \end{aligned}$$

where $b_i \in \mathbb{F}[x],\ 1 \le |a_i| < |b_i|\ (i \ge 1)$, $\ \beta_n \in \mathbf{F},\ |\beta_n| > 1$ if exists.

3 Convergence and Uniqueness

Consider a continued fraction $\xi = [b_0;\ a_1,\ b_1;\ a_2,\ b_2; \ldots]$, where $|a_n| < |b_n|$ $\forall n \ge 1$. Define

$$\begin{aligned} &A_{-1} = 1, \qquad A_0 = b_0, \qquad B_{-1} = 0, \qquad B_0 = 1, \\ &A_n = b_nA_{n-1} + a_nA_{n-2} \qquad (n = 1, 2, 3, \ldots), \\ &B_n = b_nB_{n-1} + a_nB_{n-2} \qquad (n = 1, 2, 3, \ldots), \\ &\xi_n = [0 : a_n,\ b_n;\ a_{n+1},\ b_{n+1}; \ldots] = \frac{a_n}{b_n+}\ \frac{a_{n+1}}{b_{n+1}+} \cdots. \end{aligned}$$

The next proposition summarizes basic properties of our non-regular continued fractions whose straightforward proof, see e.g. [2], is omitted.

Proposition 3.1. *(1) For $\zeta \in \mathbf{F}$, we have*

$$[b_0;\ a_1,\ b_1;\ \ldots;\ a_n,\ b_n; \zeta] = \frac{A_n + A_{n-1}\zeta}{B_n + B_{n-1}\zeta} \quad (n \ge 1).$$

(2) $[b_0;\ a_1,\ b_1;\ldots;\ a_n,\ b_n] = \dfrac{A_n}{B_n} \quad (n \geq 1)$.

(3) $A_nB_{n-1} - B_nA_{n-1} = (-1)^{n-1}\prod_{i=1}^{n} a_i \quad (n \geq 1)$.

(4) $|\xi_n| = \left|\dfrac{a_n}{b_n}\right| < 1 \quad (n \geq 1)$.

(5) $|B_{n+1}| > |B_n| \quad (n \geq 1)$

(6) $|B_n + B_{n-1}\xi_{n+1}| = |B_n| \quad (n \geq 1)$.

(7) $|B_n| = \prod_{i=1}^{n} |b_i| \quad (n \geq 1)$.

(8) $|B_n| \geq 2^n \quad (n \geq 1)$.

(9) $\dfrac{A_n}{B_n} - \xi = \dfrac{\xi_{n+1}(-1)^{n+1}\prod_{i=1}^{n} a_i}{B_n(B_n + B_{n-1}\xi_{n+1})}$.

From the properties (9), (6), (7), (4) and (8), we get

$$\left|\frac{A_n}{B_n} - \xi\right| = \frac{|\xi_{n+1}|\,|\prod_{i=1}^{n} a_i|}{|B_n||B_n + B_{n-1}\xi_{n+1}|} = \frac{|\xi_{n+1}||\prod_{i=1}^{n} a_i|}{|\prod_{i=1}^{n} b_i||B_n|} < \frac{1}{|B_n|} \leq \frac{1}{2^n} \to 0 \quad (n \to \infty),$$

which shows that our algorithm yields a convergent continued fraction expansion to each $\xi \in \mathbb{F}$. Finally to ensure uniqueness, asume that

$$b_0 + \frac{a_1}{b_1+}\ \frac{a_2}{b_2+}\cdots = \xi = b_0' + \frac{a_1}{b_1'+}\ \frac{a_2}{b_2'+}\cdots.$$

Since $b_0, b_0' \in \mathbb{F}[x]$, both $\left|\frac{a_1}{b_1+}\ \frac{a_2}{b_2+}\cdots\right|$ and $\left|\frac{a_1}{b_1'+}\ \frac{a_2}{b_2'+}\cdots\right|$ are < 1, we get

$$b_0 = [\xi] = b_0', \quad \frac{a_1}{b_1+}\ \frac{a_2}{b_2+\ldots} = (\xi) = \frac{a_1}{b_1'+}\ \frac{a_2}{b_2'+\ldots}.$$

Continue in the same manner, we get $b_i = b_i' \ \ (i \geq 0)$.

4 Characterization of Rational Elements

In this section, we show that an element in $\mathbf{F}$ is rational, i.e. belongs to $\mathbb{F}(x)$, if and only if its non-regular continued fraction is finite.

Theorem 4.1. *Let $\xi \in \mathbf{F}$, $\{a_n\}_{n=1}^{\infty}$ any sequence in $\mathbb{F}[x] \setminus \{0\}$. Then $\xi \in \mathbb{F}(x)$ if and only if its non-regular continued fraction with respect to $\{a_n\}_{n=1}^{\infty}$ is finite.*

Proof. If its non-regular continued fraction is finite, then ξ is clearly rational. Now consider $\xi \in \mathbb{F}(x)$. Let its non-regular continued fraction with respect to $\{a_n\}_{n=1}^{\infty}$ be

$$\xi = b_0 + \frac{a_1}{b_1+}\ \frac{a_2}{b_2+}\ \frac{a_3}{b_3+} \dots + \frac{a_{n-1}}{b_{n-1} + \xi_n},$$

where now ξ_n must also be rational, so that we can write

$$\xi_n = \frac{p_n}{q_n} = \frac{a_n}{a_n q_n / p_n}, \quad |\xi_n| < 1$$

where $p_n, q_n \in \mathbb{F}[x]$ are relatively prime. By the division algorithm,

$$a_n q_n = d_n p_n + e_n \ , \ |e_n| < |p_n|,$$

i.e.,

$$\frac{a_n q_n}{p_n} = d_n + \frac{e_n}{p_n} \ , \ \left|\frac{e_n}{p_n}\right| < 1.$$

Hence,

$$d_n = \left[\frac{a_n q_n}{p_n}\right] = b_n.$$

Since

$$\xi_n = \frac{a_n}{a_n q_n / p_n} = \frac{a_n}{b_n + \frac{e_n}{p_n}},$$

we have

$$\xi_{n+1} = \frac{p_{n+1}}{q_{n+1}} = \frac{e_n}{p_n}.$$

From $p_n p_{n+1} = e_n q_{n+1}$ and p_{n+1}, q_{n+1} being relatively prime, we deduce that $q_{n+1} | p_n$, and so $|q_{n+1}| \le |p_n|$. Thus,

$$1 \le |q_{n+1}| \le |p_n| < |q_n|.$$

It follows that there must exist an index n_0 such that $|q_{n_0}| = 1$ which in turn yields $\xi_{n_0} = \frac{p_{n_0}}{q_{n_0}} \in \mathbb{F}[x]$, resulting in a finite continued fraction. □

Acknowledgements: I would like to extend sincere thanks to Tuangrat Chaichana for her helpful suggestions both grammatical and mathematical. In addition, I thank Oranit Panprasitwech, Supredee Dangskul, and Jairach Promrak for their advice.

References

[1] T. Chaichana, *Continued fractions in fields of positive characteristic*, Ph.D. thesis, Chulalongkorn University, Thailand, 2001.

[2] W.B. Jones and W.J. Thron, *Continued Fractions*, Addison-Wesley Publishing Company, Inc., 1980.

Vichian Laohakosol
Department of Mathematics,
Kasetsart University,
Bangkok 10900, Thailand.
Email: `fscivil@ku.ac.th`

Panupong Vichitkunakorn
Department of Mathematics,
Prince of Songkla University,
Songkhla 90112, Thailand.
Email: `jdai_ming@hotmail.com, s4820523@psu.ac.th`