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Independence Among Various Versions of The Cauchy's Functional Equation

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Abstract: The notion of ZI-independence, introduced by Dhombres, among four versions of the Cauchy's functional equation is investigated for solution functions sending the positive real numbers into the complex numbers.

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1 Introduction

The four versions of the classical Cauchy's functional equation are

$$f(x+y) = f(x) + f(y) \tag{A}$$

$$f(xy) = f(x)f(y) \tag{M}$$

$$f(x+y) = f(x)f(y) \tag{E}$$

$$f(xy) = f(x) + f(y) \tag{L}$$

The following independence notion was first introduced by Dhombres in [3]; there he used the word s-independence instead of ZI-independence.

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Definition 1.1. Let $(\alpha), (\beta)$ be two distinct equations taken from of (A), (M), (E) and (L). The pair $\{(\alpha), (\beta)\}$ is **ZI-independent over** (X, Y) if the only common solution functions $f : X \to Y$ to (α) and (β) are either the zero function or the identity function. In the case $X = Y$, we simply say $\{(\alpha), (\beta)\}$ is **ZI-independent over** X.

In [3], Dhombres stated the following results without proof:

- The pairs of equations {(A),(E)} and {(M),(E)} are ZI-independent over a ring.
- The pairs of equations {(A),(L)} and {(M),(L)} are ZI-independent over $(\mathbb{R}^+, \mathbb{R})$ and they are a fortiori ZI-independent over $\mathbb{R}$ or $\mathbb{C}$.

We complement Dhombres's work here by investigating ZI-independence among all four versions of the Cauchy's functional equation for solution functions sending the positive real numbers into the complex field.

2 Results

We first prove some auxiliary lemmas with less restriction on the domain and codomain.

Lemma 2.1. *Let X be a set, Y an entire ring and $f : X \to Y$. Then $f \equiv 0$ and $f \equiv 2$ are the only solutions of the functional equation*

$$f(x) + f(y) = f(x)f(y). \tag{1}$$

Proof. Putting $y = x$, we obtain $2f(x) = f(x)^2$. Thus, for each $x \in X$, either $f(x) = 0$ or $f(x) = 2$.

We proceed to show that either $f \equiv 0$ or $f \equiv 2$. Suppose that there exists x_0 such that $f(x_0) = 2$. Then, $f(x) + 2 = 2f(x)$ $(x \in X)$, implying that $f \equiv 2$. □

Lemma 2.2. *Let Y be a set and $f : \mathbb{R}^+ \to Y$. If f satisfies*

$$f(xy) = f(x + y), \tag{2}$$

then f is a constant function.

Proof. Putting $y = 1$ in (2), we get $f(x) = f(x+1)$. Substituting $y+1$ for y in (2), we obtain

$$f(xy+x) = f(x(y+1)) = f(x+(y+1)) = f(x+y+1) = f(x+y) = f(xy).$$

Let z, w be distinct elements with $z > w$. Then, by above equation,

$$\begin{aligned} f(z) &= f(w+(z-w)) = f\left((z-w)(z-w)^{-1}w + (z-w)\right) \\ &= f\left((z-w)(z-w)^{-1}w\right) = f(w). \end{aligned}$$

□

Our first two main results read:

Theorem 2.3. *The pairs* $\{(\mathrm{A}), (\mathrm{E})\}, \{(\mathrm{M}), (\mathrm{L})\}$ *and* $\{(\mathrm{A}), (\mathrm{L})\}$ *are ZI-independent over* $(\mathbb{R}^+, \mathbb{C})$ *while* $\{(\mathrm{M}), (\mathrm{E})\}$ *is not.*

Proof. Observe that both pairs of equation $\{(\mathrm{A}), (\mathrm{E})\}$ and $\{(\mathrm{M}), (\mathrm{L})\}$ lead to the equation (1). It thus follows from Lemma 2.1 that $f \equiv 0$ or $f \equiv 2$. However, by direct checking, $f \equiv 2$ is not a solution of any of (A),(M),(E) or (L). Hence the pairs (A,E), and (M,L) are ZI-independent over $(\mathbb{R}^+, \mathbb{C})$.

Both pairs of equation {(M),(E)} and {(A),(L)} yield the equation (2). Hence, by Lemma 2.2, their solutions must be constant functions. Direct checking shows that $f \equiv 0$ or $f \equiv 1$ are the only solutions of the pair {(M),(E)} and $f \equiv 0$ is the only solution of the pair {(A),(L)}. Thus the pair {(A),(L)} is ZI-independent over $(\mathbb{R}^+, \mathbb{C})$ while {(M),(E)} is not. □

Theorem 2.4. *The pair* $\{(\mathrm{E}), (\mathrm{L})\}$ *is ZI-independent over* $(\mathbb{R}^+, \mathbb{C})$.

Proof. As is well-known, see e.g. [1] or [4], the equation (E) yields $f(q) = f(1)^q$ for all $q \in \mathbb{Q}^+$. Replacing x and y by 1 in (L), we obtain $f(1) = f(1) + f(1)$, which implies $f(1) = 0$ and so $f(q) = 0$ for all $q \in \mathbb{Q}^+$.

Let ζ be a positive irrational number. If $\zeta > 1$, then, by (E),

$$f(\zeta) = f(\zeta - 1 + 1) = f(\zeta - 1)f(1) = 0.$$

If $\zeta < 1$, then, by (L),

$$0 = f(1) = f\left(\zeta \frac{1}{\zeta}\right) = f(\zeta) + f\left(\frac{1}{\zeta}\right).$$

Using also the above facts, we deduce $f(\zeta) = -f(\frac{1}{\zeta}) = 0$. Therefore, f is the zero function which implies the ZI-independence of the pair {(E),(L)}. □

It is well-known that for a solution of (A) over $\mathbb{Q}$, there is a constant $c \in \mathbb{R}$ such that

$$f(x) = cx \quad (x \in \mathbb{Q}). \tag{3}$$

However, a general form of solution to (A) over $\mathbb{R}$ is much more complex; see e.g. Chapter 2 of [4]. Indeed, assuming the axiom of choice there are uncountably many non-continuous solution functions to the Cauchy's functional equation (A), a fact proved in 1905 by Georg Hamel using Hamel bases. Following the work in Chapter 2 of [4], an example of such a class of functions satisfying (A) is given by

$$f_A(x) = r_1 g(h_1) + \cdots + r_n g(h_n),$$

where H is a Hamel basis of $\mathbb{R}$, $x = r_1 h_1 + \cdots + r_n h_n$ $(r_i \in \mathbb{Q}, h_i \in H)$ is the unique representation of $x \in \mathbb{R}$ with respect to H, and g is any function defined over H. In the same manner, a particular class of uncountably many non-continuous functions satisfying (M) is given by

$$f_M(x) = \exp\left(R_1 G(h_1) + \cdots + R_n G(h_n)\right),$$

where $\log x = R_1 h_1 + \cdots + R_n h_n$ is the unique representation of $\log x$ $(x \in \mathbb{R}^+)$ with respect to the Hamel basis, H, and G is any function defined over H. It seems likely that there may be a number of common solutions to (A) and (M) and to get a meaningful result about their ZI-independence, some condition(s) may be necessary. To do so, we first note a simple lemma based on the following fact, Corollary 4 on page 15 of [2].

Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a solution of (A). If the image of f is not dense in $\mathbb{R}$, then $f(x) = cx$ for some constant c.

Lemma 2.5. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a solution of* (A) *and assume the image of f is not dense in $\mathbb{R}$.*

1. *If $f(1) = 1$, then f is the identity function.*
2. *If $f(1) = 0$, then f is the zero function.*

Proof. Using the fact just mentioned, we deduce $f(x) = cx$ for some constant c. The values $f(1) = 1$, respectively, $f(1) = 0$ yield $c = 1$, respectively, $c = 0$ ☐

Our final result reads:

Theorem 2.6. *The pair of functional equations* $\{(\mathrm{A}),(\mathrm{M})\}$ *is ZI-independent over* $(\mathbb{R}^+, K)$ *where* $\mathbb{C} \supset K = K_x + iK_y$ *and either* K_x *or* K_y *is a non-dense subset of* $\mathbb{R}$.

Proof. Let $f : \mathbb{R}^+ \to K$ be a function satisfying both (A) and (M). Putting x and y equal to 1 in (M), we obtain $f(1) = 0$ or $f(1) = 1$. If $f(1) = 0$, using (M), f is the zero function. Assume that $f(1) = 1$. In this case, we express

$$f(x) = u(x) + iv(x),$$

where u and v are real-valued functions on $\mathbb{R}^+$. Thus, $1 = f(1) = u(1) + iv(1)$, which implies

$$u(1) = 1 \text{ and } v(1) = 0. \tag{4}$$

Consequently, $u(q) = q$ and $v(q) = 0$ for all $q \in \mathbb{Q}^+$. Since either K_x or K_y is not dense in $\mathbb{R}$, either the image of u or the image of v cannot be dense in $\mathbb{R}$. We consider each case seperately.

If the image of u is not dense in $\mathbb{R}$, by Lemma 2.5 and (4), u is the identity function and so $f(x) = x + iv(x)$. By (M),

$$\begin{aligned} xy + iv(xy) &= (x + iv(x))(y + iv(y)) \\ &= xy - v(x)v(y) + i\{xv(y) + yv(x)\}, \end{aligned}$$

i.e., $v(x)v(y) = 0$ for all $x, y \in \mathbb{R}^+$. Consequently, $v \equiv 0$ which implies that f is the identity function.

If the image of v is not dense in $\mathbb{R}$, by Lemma 2.5 and (4), v is the zero function and so f is a real-valued function. By (M), for each $x \in \mathbb{R}^+$,

$$f(x) = f((\sqrt{x})^2) = f(\sqrt{x})^2 \geq 0,$$

and so the image of f is not dense in $\mathbb{R}$. Using Lemma 2.5, we obtain that f is the identity function. Therefore {(A),(M)} is ZI-independent over $(\mathbb{R}^+, K)$. □

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