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Some Classes of Finite Supersoluble Groups

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Abstract: In this survey we study the relation between the class of groups in which Sylow permutability is a transitive relation (the PST-groups) and the class of groups in which every subgroup possesses supergroups of all possible indices, the so-called $\mathcal{Y}$-groups. The parellelism between these classes in the soluble universe and the interest of the local study of PST-groups motivates a local study of $\mathcal{Y}$-groups.

A group G factorised as a product of two subgroups A and B is said to be a mutually permutable product whenever A permutes with every subgroup of B and B permutes with every subgroup of A. We present some results concerning mutually permutable products of groups in the orbit of the above classes.

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Throughout this article, all groups are finite.

The search for interesting families of subgroups in a group and the study of the way these families are embedded in the group is one of the aims of the theory of groups. One of the first results in the study of groups is the theorem of Lagrange:

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Theorem 1 (Lagrange). *If H is a subgroup of a group G, then $|G| = |H| \cdot |G : H|$.*

In particular, the order and the index of every subgroup divide the order of the group. Here we are interested in the converse of the theorem of Lagrange, namely:

Question 2. Assume that d divides $|G|$. Can we ensure the existence of a subgroup H of G of order d?

The answer is false in general, because the alternating group of degree 4 does not have subgroups of order 6. Groups in which this property holds are said to satisfy the converse of Lagrange's theorem or simply CLT-groups.

Now let p be a prime and $|G|_p$ be the largest power of p dividing $|G|$. Since p divides $|H|$ or $|G : H|$, one may consider the case in which $|G|_p = |H|$. These subgroups H are known as *Sylow p-subgroups* of G. A well-known theorem of Sylow shows that Sylow p-subgroups exist for all primes p dividing the order of the group. In fact, this theorem can be interpreted as a partial converse to Lagrange's theorem, because it implies the following:

Corollary 3. *If d is a prime-power dividing $|G|$, then G has a subgroup of order d.*

McCarthy [23] has shown in 1970 that this partial converse is in some sense the "best one" by proving the following:

Theorem 4. *Let d be any positive integer which is not a prime power. Then there is a group G such that d divides $|G|$ and G has no subgroups of order d.*

Another way to find a converse to Lagrange's theorem is asking for subgroups H such that $|G : H| = |G|_p$. Such subgroups are called *Sylow p-complements* because if H is one of these subgroups and P is a Sylow p-subgroup of G, then H is a complement to P in G, that is, $G = HP$ and $H \cap P = 1$. Unfortunately, not every group has Sylow p-complements. For instance, A_5 does not have Sylow 2-complements. In fact, Hall [17, 18] proved the following result:

Theorem 5. *A group G is soluble if and only if G has Sylow p-complements for all p.*

The theorem of Hall is a nice application of the celebrated $p^a q^b$-theorem of Burnside and opens the door to he systematic study of the class of all soluble groups. A consequence of this result is that solubility can be characterised by means of the existence of subgroups whose index and order are coprime: the so-called *Hall subgroups*.

The classes of all nilpotent and all supersoluble groups are subclasses of the class of all soluble groups which admit characterisations by the existence of subgroups. Holmes [19] showed that a group G is nilpotent if and only if G contains a normal subgroup of each possible order. Ore [24] and Zappa [29] obtained a similar characterisation for supersoluble groups:

Theorem 6. *A group G is supersoluble if and only if each subgroup $H \leq G$ contains a subgroup of order d for each divisor d of $|H|$.*

This condition does not give supersolubility when the above condition is replaced by "G contains a subgroup of every possible order": There are groups which satisfy the converse of Lagrange's theorem but are not supersoluble, like the symmetric group Σ_4 of degree 4.

Of course, we can state Theorem 6 in the following equivalent way, more easily treated:

Theorem 7. *A group G is supersoluble if and only if each subgroup $H \leq G$ contains a subgroup of index p for each prime divisor p of $|H|$.*

The condition on a group G given in Theorem 7, namely

> for all $H \leq G$ and for all primes q dividing $|H|$, there exists a subgroup K of G such that $K \leq H$ and $|H : K| = q$,

has a dual formulation:

> for all $H \leq G$ and for all primes q dividing $|G : H|$, there exists a subgroup K of G such that $H \leq K$ and $|K : H| = q$.

Some authors have studied the groups satisfying the latter condition. We will call them $\mathcal{Y}$-groups, like in [26, Chapter 1, 4].

Since every maximal subgroup of a $\mathcal{Y}$-group has prime index, the class $\mathcal{Y}$ is a subclass of the class of all supersoluble groups. However, the containment is proper:

Example 8. Let $G = \langle x, y, z \mid x^3 = z^3 = y^2 = (xy)^2 = 1, xz = zx, yz = zy \rangle$. Then G is the direct product of $\langle x, y \rangle$, which is isomorphic to a symmetric group of degree 3, and $\langle z\rangle$, which is a cyclic group of order 3. Consider the subgroup $H = |xz|$. Then $|G : H| = 6$, but G does not contain any subgroup K such that $|K : H| = 2$. Therefore G is not a $\mathcal{Y}$-group. However, it is clear that G is supersoluble.

The class of $\mathcal{Y}$-groups admit quite nice characterisations (see [23]):

Theorem 9. *Let G be a group.*

1. *$G \in \mathcal{Y}$ if and only if every subgroup of G can be written as an intersection of subgroups of G of coprime prime-power indices.*
2. *(Deskins-Venzke) A group $G \in \mathcal{Y}$ if and only if the nilpotent residual L of G is a nilpotent Hall subgroup of G and $G = L\mathrm{N}_G(H)$ for every $H \leq L$.*

Despite these nice characterisations, the class $\mathcal{Y}$ is not closed under taking subgroups.

Example 10. Let $X = \langle a, b, c \mid a^3 = b^3 = c^3 = 1, [a, b] = c, ca = ac, bc = cb\rangle$ be the nonabelian group of order 27 of exponent 3. Then X has an automorphism t of order 2 such that $a^t = a^{-1}$, $b^t = b^{-1}$, and $c^t = c$. Let $G = [X]\langle t\rangle$ be the semidirect product of X by $\langle t\rangle$. Then G has order 54, $\Phi(G) = \mathrm{Z}(G) = \langle c\rangle$, and G is a $\mathcal{Y}$-group. Also note that $\langle a, t\rangle \cong S_3$, so that $\langle a, t\rangle \times \langle c\rangle$ is a subgroup of G which is isomorphic to $S_3 \times C_3$ and hence is not a $\mathcal{Y}$-group.

The second characterisation shows that the class $\mathcal{Y}$ is closely related to the class of all PST-groups, or groups in which every subnormal subgroup permutes with all Sylow subgroups.

A subgroup K of G is said to be *permutable* (respectively, *S-permutable*) in G provided $KH = HK$ for all subgroups (respectively, Sylow subgroups) H of G. A well known result of Ore [24] shows that permutable subgroups are necessarily subnormal. Kegel [22] generalised this result showing that S-permutable subgroups are subnormal. A group G is called a T*-group* provided normality is a transitive relation, that is, H normal in K and K normal in G implies that H is a normal subgroup of G. Similarly, one defines PT*-groups* and PST*-groups* as those groups in which, respectively, permutability and Sylow-permutability are transitive relations, respectively. As a consequence of the result of Ore (respectively, Kegel), it follows that PT-groups (respectively, PST-groups) are those groups in which permutability (respectively, Sylow-permutability) coincides with subnormality. The classes T, PT, and PST have been extensively investigated with many characterisations available. Much of these characterisations have been obtained in the past 10 years.

Let us recall the classical theorem of Agrawal [1]:

Theorem 11. *A group G is a soluble PST-group if and only if G has an abelian normal Hall subgroup N of odd order such that G/N is nilpotent and the elements of G induce power automorphisms in N.*

If we add in this result "G/N nilpotent modular group," we obtain the characterisation of soluble PT-groups given by Zacher [28], and if we put "G/N Dedekind," we get Gaschütz's characterisation of soluble T-groups [16].

A consequence of the Deskins and Venzke theorem, Theorem 11, Gaschütz's characterisation, and Dedekind's theorem [20, III, 7.12] is:

Theorem 12. *Let G be a group.*

1. *If G is a soluble PST-group, then G is a $\mathcal{Y}$-group.*
2. *Assume that $G \in \mathcal{Y}$. Then G is a soluble PST-group if and only if the nilpotent residual of G is abelian.*
3. *Assume that $G \in \mathcal{Y}$. Then G is a soluble T-group if and only if all Sylow subgroups of G are Dedekind.*

The classical characterisations show that the classes T, PT, and PST are subgroup-closed in the soluble universe and also shows that the difference between them is the Sylow structure. Hence local techniques turn out to be useful in the study of these classes.

A class of groups $\mathcal{A}$ is said to be "local" if it is generalised in a form referring to a prime, $\mathcal{A}_p$ in such a way the original class can be described as the conjunction of all the local classes for all primes. For instance, p-nilpotence, p-supersolubility, and p-solubility are the local versions of nilpotency, supersolubility, and solubility, respectively.

For a prime p, Bryce and Cossey [15] defined the class T_p of all soluble groups G for which every subnormal p'-perfect subgroup of G is normal. They proved:

Theorem 13. *A soluble group is a T-group if and only if it is a T_p-group for all primes p.*

In [3], Alejandre, the first author, and Pedraza-Aguilera introduced in the soluble universe the class PST_p of all soluble groups G in which every p'-perfect subnormal subgroup in G permutes with every Hall p'-subgroup of G. This condition is equivalent to G being p-supersoluble and having all its p-chief factors isomorphic when regarded as modules over G (see [3]). This result not only holds in the soluble universe, but also in the p-soluble one.

Theorem 14. *A soluble group G is a* PST*-group if and only if G is a* PST_p*-group for all primes p.*

Beidleman and Heineken defined in [12] the class T''_p, for a prime p, of all soluble groups G in which every p'-perfect subnormal subgroup of G is S-permutable in G and proved:

Theorem 15. *A soluble group G is a* PST*-group if and only if it is a* T''_p*-group for all primes p.*

A similar result holds for PT-groups replacing S-permutability by permutability. In [9, Theorem A], the following local version of Agrawal's result was obtained. For each group X and every prime p, $X(p)$ denotes the p-nilpotent residual of X, that is, the smallest normal subgroup N of X such that X/N is p-nilpotent, while $\mathrm{O}_{p'}(X)$ denotes the largest normal p'-subgroup of X.

Theorem 16. *A p-soluble group G is a* PST_p*-group if, and only if, one of the following two conditions holds:*

1. *G is p-nilpotent, or*
2. *the subgroup $G(p)/\mathrm{O}_{p'}\big(G(p)\big)$ is an abelian normal Sylow p-subgroup of the group $G/\mathrm{O}_{p'}\big(G(p)\big)$ in which the elements of $G/\mathrm{O}_{p'}\big(G(p)\big)$ induce power automorphisms.*

Theorem 11 follows from Theorems 14 and 16, as shown in [9]

Bearing in mind that these local approaches and the relationship between T-groups and PST-groups, it is natural to ask for local versions of the class of $\mathcal{Y}$-groups. It was accomplished in [6].

Definition 17. A group G *satisfies the property* $\mathcal{Z}_p$ when for every p-subgroup X of G and for every power of a prime q, q^m, dividing $|G : X\mathrm{O}_{p'}(G)|$, there exists a subgroup K of G containing $X\mathrm{O}_{p'}(G)$ such that $|K : X\mathrm{O}_{p'}(G)| = q^m$.

Definition 18. Let G be a group. We say that G *satisfies* $\mathcal{Z}'_p$ if G satisfies either of the following conditions:

1. G is p-nilpotent, or
2. $G(p)/\operatorname{O}_{p'}\bigl(G(p)\bigr)$ is a Sylow p-subgroup of $G/\operatorname{O}_{p'}\bigl(G(p)\bigr)$ and for every p-subgroup H of $G(p)$, we have that $G = G(p)\operatorname{N}_G(H)$.

Our next result can be regarded as the analogue of Theorem 16:

Theorem 19 ([6, Theorem 13]). *Let G be a p-soluble group. Then G satisfies $\mathcal{Z}_p$ if and only if G satisfies $\mathcal{Z}'_p$.*

We are now in a position to show that the class $\mathcal{Y}$ is a local class:

Theorem 20 ([6, Theorem 15]). *A group G satisfies $\mathcal{Y}$ if and only if G satisfies $\mathcal{Z}_p$ for all primes p.*

Assume now that a group G can be factorised as $G = G_1G_2\cdots G_m$ which is a product of some pairwise permutable subgroups. A natural question in this context is: What can be said about G if some properties of the factors G_i are known? For instance, a well-known theorem of Kegel and Wielandt [21, 27] says that a product of two nilpotent groups is soluble. The fact that a product of two supersoluble groups is not necessarily supersoluble, even if both factors are normal in the group, but a direct product of supersoluble groups is supersoluble, motivates the restriction of this question to factorised groups in which both factors are connected by certain permutability properties which are stronger than the simple permutability between the factors, but weaker than the centralisation of the elements of the factors in the direct product. The first author and Shaalan introduced in [5] the notion of *mutually permutable product* $G = AB$ of two subgroups A and B: in a mutually permutable product, each factor permutes with every subgroup of the other factor. In particular, this situation holds when both factors are normal in the group. Some results about normal products of supersoluble groups were extended to mutually permutable products in [5], for instance, a mutually permutable product $G = AB$ of two supersoluble groups A and B is supersoluble whenever G' is nilpotent or one of the factors is nilpotent. They also showed that totally permutable products (that is, products in which every subgroup of each factor permutes with every subgroup of the other factor) of supersoluble groups are supersoluble. Of course, central products and direct products are instances of totally permutable products. Mutually and totally permutable products have been considered as well in [2, 8, 10, 11, 13, 14].

As a consequence of the theorem of Agrawal [1], soluble PST-groups are supersoluble. Robinson [25] showed that, in the general finite universe, PST-groups have all their chief factors simple, or, as he says, they are *SC-groups*. The classification of finite simple groups and the truth of the Schreier conjecture yield the following description of SC-groups:

Theorem 21 ([25, Proposition 2.4]). *A group G is an SC-group if and only if there is a perfect normal subgroup D such that G/D is supersoluble, $D/\mathrm{Z}(D)$ is a direct product of G-invariant simple groups, and $\mathrm{Z}(D)$ is supersolubly embedded in G (i.e., there is a G-admissible series of $\mathrm{Z}(D)$ with cyclic factors).*

The relation between totally and mutually permutable products and SC-groups has been investigated in [7, 8, 10, 13, 14]. For instance:

Theorem 22 ([8, Theorems 2 and 3]). *Assume that G is the mutually permutable product of its subgroups A and B. Then:*

1. *If G is an SC-group, then A and B are SC-groups.*
2. *If A and B are SC-groups, then $G/\operatorname{Core}_G(A \cap B)$ is an SC-group.*

In [4] we prove some results on mutually permutable products whose factors belong to some of the above classes. We start with a localisation of SC-groups.

Definition 23. Let p be a prime number. We say that a group G is an *SC_p-group* whenever every chief factor of G whose order is divisible by p is simple.

It is clear that G is an SC-group (i.e., all its chief factors are simple) if and only if G is and SC_p-group for all primes p. In what follows, p will denote a fixed prime number. The proofs of Theorem 22 can be adapted to prove:

Lemma 24 ([4, Lemma 11]). *Assume that G is a mutually permutable product of its subgroups A and B.*

1. *If G is an SC_p-group, then A and B are SC_p-groups.*
2. *If A and B are SC_p-groups, then $G/\operatorname{Core}_G(A \cap B)$ is an SC_p-group.*

Mutually permutable products of SC_p-groups and p-soluble $\mathcal{Z}_p$-groups are the object of the next result:

Theorem 25 ([4, Theorem 12]). *Let $G = AB$ be a mutually permutable product of its subgroups A and B. Assume that A is an SC_p-group and that B is a p-soluble $\mathcal{Z}_p$-group. Then G is an SC_p-group.*

The following corollaries follow immediately from Theorem 25:

Corollary 26 ([4, Corollary 13]). *If G is a mutually permutable product of an SC-group A and a $\mathcal{Y}$-group B, then G is an SC-group. In particular, if G is a mutually permutable product of a supersoluble group A and a $\mathcal{Y}$-group B, then G is supersoluble.*

Let $\mathfrak{X}$ be a class of groups. A class of groups $\mathfrak{F}$ is called the *Fitting core* of $\mathfrak{X}$ provided that whenever if $A \in \mathfrak{X}$ and $B \in \mathfrak{F}$, and A and B are normal subgroups of a group G, then $AB \in \mathfrak{X}$ (see [12]). From Corollary 2 of [12] it follows that the class of soluble PST-groups belongs to the Fitting core of the formation of supersoluble groups. In fact from Corollary 26 we obtain a more general statement, mainly: the class $\mathcal{Y}$ is contained in the Fitting core of both the formation of supersoluble groups and, hence, the formation of SC-groups.

Corollary 27 ([4, Corollary 14]). *If G is a mutually permutable product of two p-soluble $\mathcal{Z}_p$-groups, then G is p-supersoluble.*

Corollary 28 ([4, Corollary 15]). *If G is a mutually permutable product of two $\mathcal{Y}$-groups, then G is supersoluble.*

Corollary 28 admits the following generalisation:

Theorem 29. *Let $G = G_1G_2\cdots G_r$ be a group such that G_1, $G_2, \dots$, G_r are pairwise mutually permutable subgroups of G. If all G_i are $\mathcal{Y}$-groups, then G is supersoluble.*

In [14, Theorem 5], the following result is proved:

Theorem 30. *Let $G = AB$ be a mutually permutable product of the subgroups A and B. If G is a PST-group, then A is a PST-group.*

Our proof of Theorem 32 depends on the following:

Lemma 31 ([4, Lemma 20]). *Let N be a normal subgroup of G such that G/N satisfies N_p. If either N is non-abelian and simple or N is a p'-group, then G satisfies N_p.*

We conclude with a local version of Theorem 30, from which it follows immediately:

Theorem 32 ([4, Theorem 18]). *Let G be a mutually permutable product of its subgroups A and B. If G is a* SC*-group and satisfies N_p, then A satisfies N_p.*

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