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Normal Ideals of Pseudo-Complemented Distributive Lattices

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Abstract: The notion of normlets and normal ideals are introduced in a pseudo-complemented distributive lattice and then normal ideals are characterized in terms of normlets. Some equivalent conditions are derived for a pseudo-complemented lattice to become a disjunctive lattice. The properties of direct products of normal ideals are studied. A set of equivalent conditions is derived for every prime normal ideal to become a minimal prime ideal in topological terms.

Keywords: Pseudo-complemented distributive lattice, normal ideal, normlet, direct product, minimal prime ideal, Hausdorff space.

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1 Introduction

Distributive lattices form one of the most interesting classes of lattices. Lattices, especially distributive lattices and Boolean algebras, arise naturally in logic, and thus some of the elementary theory of lattices had been worked out earlier by many mathematicians. G. Gratzer [3] investigated some of the interesting properties of distributive lattices. The theory of pseudo-complements in lattices, and particularly in distributive lattices was developed by M.H. Stone [1], O. Frink [2], and G. Gratzer [3]. Later many authors like R. Balbes [4], O. Frink [2]etc.,

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extended the study of pseudo-complements to characterize Stone lattices. W.H. Cornish [5] and T.P. Speed [6] studied some properties of congruences in distributive lattices. In 2015, the author [8] introduced the notion of disjunctive ideals of distributive lattices and characterized disjunctive ideals. Recently in 2017, the author [9] introduced the concepts of coaxer ideals and coaxer lattices of distributive lattices and characterized Boolean algebras in terms of coaxer ideals and congruences.

The main aim of this paper is to study the properties of normal ideals and to observe the properties of prime normal ideals in topological sense. In this paper, the concept of normlets is introduced and normal ideals are then characterized in terms of normlets. A property of the cartesian product of normal ideals is obtained. A set of equivalent conditions is derived for an ideal to become a normal ideal which leads to a characterization of disjunctive lattices. It is observed that every minimal prime ideal is normal ideal. Though the converse of the above is not true, a set of equivalent conditions are derived for every prime normal ideal to become a minimal prime ideal.

A pseudo-complement of an element is a generalization of the notion of complement in a lattice. In a lattice L, an element $b \in L$ is said to have *pseudo-complement* if there exists a greatest element $b^* \in L$, disjoint from b, with the property that $b \wedge b^* = 0$. More formally

$$b^* = \max\{y \in L \mid b \wedge y = 0\}$$

The lattice L itself is called *pseudo-complemented* if every element of L is pseudo-complemented. Every pseudo-complemented lattice is necessarily bounded, i.e. it has the unit 1 as well. Since the pseudo-complement is unique by definition (if it exists), a pseudo-complemented lattice can be endowed with a unary operation $*$ mapping every element to its pseudo-complement.

A distributive lattice L in which every element has a pseudo-complement is called a *pseudo-complemented distributive lattice*. For any two elements a, b of a pseudo-complemented lattice, we have the following.

(1) $a \leq b$ implies $b^* \leq a^*$
(2) $a \leq a^{**}$
(3) $a^{***} = a^*$
(4) $(a \vee b)^* = a^* \wedge b^*$
(5) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

An element a of a pseudo-complemented lattice L is called *dense* if $a^* = 0$ and the set D of all dense element of L forms a filter in L. A pseudo-complemented lattice L is called *disjunctive* if for any $x, y \in L, x^* = y^*$ implies $x = y$. Throughout this note, L stands for a pseudo-complemented distributive lattice $(L, \vee, \wedge, ^*, 0, 1)$, unless otherwise mentioned.

For any $a \in L$, we have the principal ideal $(a]$ generated by the element a is the set $\{x \in L \mid x \leq a\}$. For the basic facts concerning minimal prime ideals, we refer to [7]. A prime ideal of L is a minimal prime ideal if it is the minimal element of the poset of all prime ideals ordered by set inclusion. A prime ideal P of L is minimal if and only if to each $x \in P$, there exists $y \notin P$ such that $x \wedge y = 0$. The set of all minimal prime ideals of L is denoted by μ_L. All hulls $h(.)$ and kernels $k(.)$ refer to μ_L. Equip $\mu = \mu_L$ with the topology induced by the closure operator $\mathcal{A} \to \mathcal{A}^- = h(k(\mathcal{A}))$. The resulting space is denoted by $Minp\ (L)$ and called the minimal spectrum of L.

Proposition 1.1. *[10] For any ideal J of L, the set $\mu(J) = \{M \in \mu \mid J \nsubseteq M\}$ is open in $Minp\ (L)$. Further, the family $\mu_L = \{\mu(x) \mid x \in L\}$ where $\mu(x) = \mu(\{x\})$, forms a (Join) basis for the open sets of $Minp\ (L)$.*

Corollary 1.2. *[10] For any ideal J of L, the set $h(J) = \{M \in \mu \mid J \subseteq M\}$ is closed in $Minp\ (L)$. Further, the family $\{h(x) \mid x \in L\}$ forms a (meet) basis for the closed sets of $Minp\ (L)$.*

Proposition 1.3. *[10] $Minp\ (L)$ is a Hausdorff space.*

The reader is refered to [3] for the notions and notations. However, some of the preliminary definitions and results are presented for the ready reference of the reader. Throughout the rest of this note all lattices are bounded and pseudo-complemented distributive lattices.

2 Characterization of normal ideals

In this section, the concept of normal ideals is introduced and characterized in a pseudo-complemented distributive lattice. Finally, a set of properties of a space of prime normal ideals is studied.

Definition 2.1. For any ideal I of a pseudo-complemented distributive lattice L, define the set I° as follows:

$$I^{\circ} = \{x \in L \mid x \wedge a^* = 0 \text{ for some } a \in I\}$$

Lemma 2.2. *For any ideal of I of L, I° is an ideal of L.*

Proof. Clearly $0 \in I^{\circ}$. Let $x, y \in I^{\circ}$. Then $x \wedge a^* = 0$ and $y \wedge b^* = 0$ for some $a, b \in I$. Hence $(x \vee y) \wedge (a \vee b)^* = (x \vee y) \wedge (a^* \wedge b^*) = (x \wedge a^* \wedge b^*) \vee (y \wedge a^* \wedge b^*) = 0$. Thus $x \vee y \in I^{\circ}$. Let $x \in I^{\circ}$ and $y \leq x$. Then we get $y \wedge a^* \leq x \wedge a^* = 0$ for some $a \in I$. Hence $y \in I^{\circ}$. Therefore I° is an ideal of L. □

We observe some basic properties of I°, in the following lemma.

Lemma 2.3. *For any two ideals I, J of L, we have the following:*

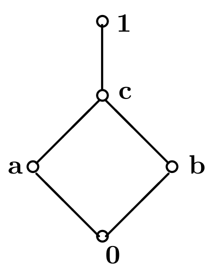
(1) $I \subseteq I^{\circ}$
(2) $I \subseteq J$ *implies* $I^{\circ} \subseteq J^{\circ}$
(3) $I^{\circ} \cap J^{\circ} = (I \cap J)^{\circ}$
(4) $(I)^{\circ\circ} = I^{\circ}$

Proof. (1). For any $a \in I$, we have $a \wedge a^* = 0$. Hence $a \in I^{\circ}$. Thus $I \subseteq I^{\circ}$.
(2). Suppose $I \subseteq J$. Let $x \in I^{\circ}$. Then $x \wedge a^* = 0$ for some $a \in I \subseteq J$. Hence $x \in J^{\circ}$.
(3). Clearly $(I \cap J)^{\circ} \subseteq I^{\circ} \cap J^{\circ}$. Conversely, let $x \in I^{\circ} \cap J^{\circ}$. Then $x \wedge a^* = 0$ and $x \wedge b^* = 0$ for some $a \in I$ and $b \in J$. Since $x \wedge a^* = 0$ and $x \wedge b^* = 0$, we get $x \leq a^{**}$ and $x \leq b^{**}$. Thus $x \leq a^{**} \wedge b^{**} = (a \wedge b)^{**}$ and $a \wedge b \in I \cap J$. Therefore $x \in (I \cap J)^{\circ}$.
(4). Clearly $I^{\circ} \subseteq I^{\circ\circ}$. Conversely, let $x \in I^{\circ\circ}$. Then $x \wedge a^* = 0$ for some $a \in I^{\circ}$. Hence $a \wedge b^* = 0$ for some $b \in I$. Hence $x \leq a^{**} \leq b^{**}$. Thus $x \wedge b^* \leq b^{**} \wedge b^* = 0$. Therefore $x \in I^{\circ}$. □

We now introduce the concept of normal ideals in the following:

Definition 2.4. An ideal I of L is called a *normal ideal* if $I = I^{\circ}$.

Example 2.5. Consider the following distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given by:



Consider the ideals $I = \{0, a\}$ and $J = \{0, a, b, c\}$. It can be easily observed that $I^\circ = \{0, a\} = I$ but $J^\circ = L$. Hence I is a normal ideal but J is not a normal ideal of L.

Proposition 2.6. *Every minimal prime ideal of L is a normal ideal.*

Proof. Let P be a minimal prime ideal of L. We always have $P \subseteq P^\circ$. Let $x \in P^\circ$. Then $x \wedge a^* = 0 \in P$ for some $a \in P$. Suppose $a^* \in P$. Then $a \vee a^* \in P$. Since P is a minimal prime ideal, there exists $y \notin P$ such that $y \wedge (a \vee a^*) = 0$. Hence $y \leq (a \vee a^*)^* = a^* \wedge a^{**} = 0 \in P$. Thus $y \in P$, which is a contradiction. Therefore $a^* \notin P$. Hence $x \in P$. Thus $P = P^\circ$. Therefore P is a normal ideal. □

Definition 2.7. An ideal of the form $(a]^\circ = \{x \in L \mid x \wedge a^* = 0\}$ is called a *normlet*. For simplicity, we denote $(a]^\circ$ by $(a)^\circ$.

We first observe some basic properties of normlets the following:

Lemma 2.8. *For any $a, b \in L$, we have the following:*

(1) $a \in (a)^\circ$
(2) $a \leq b$ *implies* $(a)^\circ \subseteq (b)^\circ$
(3) $(a)^\circ \cap (b)^\circ = (a \wedge b)^\circ$
(4) $(a)^{\circ\circ} = (a)^\circ$
(5) $a \in (b)^\circ$ *implies* $(a)^\circ \subseteq (b)^\circ$
(6) $a \in D$ *if and only if* $(a)^\circ = L$
(7) $a \vee b = 1$ *implies* $(a)^\circ \vee (b)^\circ = L$
(8) $a^* = b^*$ *if and only if* $(a)^\circ = (b)^\circ$

Proof. (1), (2), (3) and (4) are clear by Lemma 2.3.
(5). Let $a \in (b)^\circ$. Then $a \wedge b^* = 0$. Hence $b^* \leq a^*$. Let $x \in (a)^\circ$. Then $x \wedge b^* \leq x \wedge a^* = 0$. Hence $x \in (b)^\circ$. Therefore $(a)^\circ \subseteq (b)^\circ$.
(6). Let $a \in D$. Then $(a)^\circ = \{x \in L \mid x \wedge a^* = 0\} = \{x \in L \mid x \wedge 0 = 0\} = L$. Conversely, suppose $(a)^\circ = L$. Then $1 \in (a)^\circ$. Therefore $a^* = 1 \wedge a^* = 0$. Hence $a \in D$.
(7). Let $a \vee b = 1$. Suppose $(a)^\circ \vee (b)^\circ \neq L$. Then there exists a maximal ideal M such that $(a)^\circ \vee (b)^\circ \subseteq M$. Then $a \in (a)^\circ \subseteq M$ and $b \in (b)^\circ \subseteq M$. Hence $1 = a \vee b \in M$, which is a contradiction. Therefore $(a)^\circ \vee (b)^\circ = L$.
(8). Suppose $a^* = b^*$. Then clearly $(a)^\circ = (b)^\circ$. Conversely, suppose that $(a)^\circ = (b)^\circ$. Then $a \in (a)^\circ = (b)^\circ$ and hence $a \wedge b^* = 0$. Thus $b^* \leq a^*$. Similarly, we can get $a^* \leq b^*$. Therefore $a^* = b^*$. □

Clearly each normlet is a normal ideal. Let us denote the set of all normlets of L by $\mathcal{A}^\circ(L)$. Then, in the following, we prove that $\mathcal{A}^\circ(L)$ forms a Boolean algebra (complemented and distributive lattice).

Theorem 2.9. *The class $\mathcal{A}^\circ(L)$ of all normlets of L forms a Boolean algebra.*

Proof. For any $a, b \in L$, define

$$(a)^\circ \cap (b)^\circ = (a \wedge b)^\circ \quad \text{and} \quad (a)^\circ \sqcup (b)^\circ = (a \vee b)^\circ$$

Clearly $(a \wedge b)^\circ$ is the infimum of $(a)^\circ$ and $(b)^\circ$ in $\mathcal{A}^\circ(L)$. Also $(a \vee b)^\circ$ is an upper bound of $(a)^\circ$ and $(b)^\circ$. Suppose $(c)^\circ$ is an upper bound of both $(a)^\circ$ and $(b)^\circ$. Then $a \in (a)^\circ \subseteq (c)^\circ$ and $b \in (b)^\circ \subseteq (c)^\circ$. Since $(c)^\circ$ is an ideal, we get $a \vee b \in (c)^\circ$. Hence $(a \vee b)^\circ \subseteq (c)^\circ$. Therefore $(a \vee b)^\circ$ is the supremum of $(a)^\circ$ and $(b)^\circ$. It can be easily observed that $\langle \mathcal{A}^\circ(L), \cap, \sqcup, (0)^\circ, (1)^\circ \rangle$ is a bounded distributive lattice. Now for any $a \in L$, $(a)^\circ \cap (a^*)^\circ = (a \wedge a^*)^\circ = (0)^\circ = \{0\}$. Also $(a)^\circ \sqcup (a^*)^\circ = (a \vee a^*)^\circ = L$. Therefore $\langle \mathcal{A}^\circ(L), \cap, \sqcup, (0)^\circ, (1)^\circ \rangle$ is a Boolean algebra. □

We now characterize the normal ideals in terms of normlets.

Theorem 2.10. *For any ideal I of L, the following are equivalent.*

(1) I *is normal;*
(2) *For any* $a \in L, a \in I$ *implies* $(a)^\circ \subseteq I$;
(3) *For any* $a, b \in L$, $a^* = b^*$ *and* $a \in I$ *imply that* $b \in I$;
(4) *For any* $a, b \in L$, $(a)^\circ = (b)^\circ$ *and* $a \in I$ *imply that* $b \in I$;
(5) $I = \bigcup_{a \in I} (a)^\circ$.

Proof. $(1) \Rightarrow (2)$: Assume that I is normal. Let $a \in I = I^\circ$. Then $a \wedge b^* = 0$ for some $b \in I$. Let $x \in (a)^\circ$. Then $x \wedge b^* \leq x \wedge a^* = 0$. Hence $x \in I^\circ = I$. Therefore $(a)^\circ \subseteq I$.
$(2) \Rightarrow (3)$: Assume the condition (2). Let $a^* = b^*$. Then by Lemma 2.8(8), we get that $(a)^\circ = (b)^\circ$. Suppose $a \in I$. Therefore $b \in (b)^\circ = (a)^\circ \subseteq I$.
$(3) \Rightarrow (4)$: It is clear.
$(4) \Rightarrow (5)$: Assume the condition (4). For any $a \in I$, we have $(a] \subseteq (a)^\circ$. Hence $I = \bigcup_{a \in I} (a] \subseteq \bigcup_{a \in I} (a)^\circ$. Conversely, let $a \in I$ and $x \in (a)^\circ$. Hence $(x)^\circ \subseteq (a)^\circ$. Hence $(x)^\circ = (x)^\circ \cap (a)^\circ = (x \wedge a)^\circ$. Since $x \wedge a \in I$, by the condition (4), we get that $x \in I$. Thus $(a)^\circ \subseteq I$. Therefore $\bigcup_{a \in I} (a)^\circ \subseteq I$.
$(5) \Rightarrow (1)$: Assume the condition (5). Clearly $I \subseteq I^\circ$. Let $x \in I^\circ$. Then $x \wedge a^* = 0$

for some $a \in I$. Hence $x \in (a)^{\circ}$ for some $a \in I$. Therefore $x \in \bigcup_{a\in I}(a)^{\circ} = I$. Therefore I is a normal ideal. □

Let J be an ideal of a lattice L. For any $x, y \in L$, define a relation $\theta(J)$ such that $(x,y) \in \theta(J)$ if and only if $x \wedge a^* = y \wedge a^*$ for some $a \in J$. Using the properties of a pseudo-complementation in L, it can be easily obtained that $\theta(J)$ is a congruence on L. In the following theorem, the class of all normal ideals are characterized with the help of congruences.

Theorem 2.11. *Let J be an ideal in a pseudo-complemented lattice L. Then J is a normal ideal if and only if $J = Ker\ \theta(J)$.*

Proof. Assume that J is a normal ideal of L. Since $x \wedge x^* = 0$, we get that $J \subseteq Ker\ \theta(J)$. On the other hand, let $x \in Ker\ \theta(J)$. Then $x \wedge a^* = 0$ for some $a \in J = J^{\circ}$. Hence there exists some $b \in J$ such that $a \wedge b^* = 0$. It concludes that $b^* \leq a^* \leq x^*$ and hence $x \wedge b^* = 0$. Thus we get $x \in J^{\circ} = J$. Therefore $J = Ker\ \theta(J)$.

Conversely, assume that $J = Ker\ \theta(J)$. Let $x, y \in L$ be such that $x^* = y^*$ and $x \in J$. Since $x \in Ker\ \theta(J)$, we get $x \wedge a^* = 0$ for some $b \in J$. Hence $a^* \leq x^* = y^*$. Thus $y \wedge a^* = 0$ and $a \in J$. Hence $y \in Ker\ \theta(J) = J$. Therefore J is a normal ideal. □

Let L_1 and L_2 be two pseudo-complemented distributive lattices with $*$ as their pseudo-complementation. Then $L_1 \times L_2$ is also a pseudo-complemented distributive lattice with respect to the point-wise operations in which the pseudo-complementation given by:

$$(a,\ b)^* = (a^*,\ b^*)$$

It can be easily observed that the set of all normlets of $L_1 \times L_2$ forms a distributive lattice with respect to the operations of the Theorem 1.9, when considered point-wise. We now discuss about the direct products of normal ideals of a pseudo-complemented distributive lattice. We first observe the following result related to the normlets of $L_1 \times L_2$. The proof of the following lemma is straightforward.

Lemma 2.12. *Let L_1 and L_2 be two pseudo-complemented distributive lattices with $*$ as their pseudo-complementation. For any $a \in L_1$ and $b \in L_2$, we have $(a,b)^{\circ} = (a)^{\circ} \times (b)^{\circ}$.*

Theorem 2.13. *If I_1 and I_2 are normal ideals of L_1 and L_2 respectively, then $I_1 \times I_2$ is a normal ideal of the product lattice $L_1 \times L_2$. Conversely, every normal ideal of $L_1 \times L_2$ can be expressed as $I = I_1 \times I_2$ where I_1 and I_2 are normal ideals of L_1 and L_2 respectively.*

Proof. Suppose I_1 and I_2 are two normal ideals of L_1 and L_2 respectively. Then clearly $I_1 \times I_2$ is an ideal of $L_1 \times L_2$. Let $(a,b) \in I_1 \times I_2$. Then $a \in I_1$ and $b \in I_2$. Since I_1 and I_2 are normal ideals, we get $(a)^\circ \subseteq I_1$ and $(b)^\circ \subseteq I_2$. By Lemma 2.12, it now yields $(a,b)^\circ = (a)^\circ \times (b)^\circ \subseteq I_1 \times I_2$. Therefore $I_1 \times I_2$ is a normal ideal of $L_1 \times L_2$.

Conversely, let I be a normal ideal of $L_1 \times L_2$. Consider $I_1 = \{a \in L_1 \mid (a,b) \in I \text{ for some } b \in L_2\}$. Clearly I_1 is an ideal of L_1. Let $x \in I_1$. Then $(x,y) \in I$ for some $y \in L_2$. Since I is a normal ideal in $L_1 \times L_2$, we get $(x)^\circ \times (y)^\circ = (x,y)^\circ \in I$. Now, let $t \in (x)^\circ$. Since $y \in (y)^\circ$, we get $(t,y) \in (x)^\circ \times (y)^\circ \subseteq I$. Hence $t \in I_1$, which implies that $(x)^\circ \subseteq I_1$. Therefore I_1 is a normal ideal of L_1. Similarly, we can get $I_2 = \{b \in L_2 \mid (a,b) \in I \text{ for some } a \in L_1\}$ is a normal ideal of L_2.

We now prove that $I = I_1 \times I_2$. Clearly $I \subseteq I_1 \times I_2$. Conversely, let $(a,b) \in I_1 \times I_2$. Then $(a,y), (x,b) \in I$ for some $x \in L_1$ and $y \in L_2$. Since I is a normal ideal of $L_1 \times L_2$, we get

$$\begin{aligned} (a,y)^\circ \subseteq I,\ (x,b)^\circ \subseteq I \quad &\Rightarrow \quad (a,y)^\circ \sqcup (x,b)^\circ \subseteq I \\ &\Rightarrow \quad ((a,y) \vee (x,b))^\circ \subseteq I \\ &\Rightarrow \quad (a \vee x, y \vee b)^\circ \subseteq I \\ &\Rightarrow \quad (a,b)^\circ \subseteq I \\ &\Rightarrow \quad (a,b) \in I \end{aligned}$$

Hence $I_1 \times I_2 \subseteq I$. Therefore $I = I_1 \times I_2$. □

In the following theorem, a set of equivalent conditions is derived for a lattice to become disjunctive.

Theorem 2.14. *The following conditions are equivalent in a lattice.*

(1) L is a disjunctive lattice;
(2) every ideal is a normal ideal;
(3) every principal ideal is a normal ideal;
(4) every proper ideal contains no dense element;
(5) every prime ideal is a normal ideal.

Proof. (1) $\Rightarrow$ (2): Assume that L is disjunctive. Let I be an ideal of L. Choose $x, y \in L$ be such that $x^* = y^*$ and $x \in I$. Since L is disjunctive, we get $x = y$. Hence $y \in I$, which implies that I is a normal ideal.

(2) $\Rightarrow$ (3): It is obvious.

(3) $\Rightarrow$ (4): Assume that every principal ideal is a normal ideal. Let I be a proper ideal. Suppose $I \cap D \neq \emptyset$. Choose $x \in I \cap D$. Then by (3), $(x]$ is a normal ideal. Hence $(x)^\circ \subseteq (x] \subseteq I$. Since $x^{**} \wedge x^* = 0$, we get $1 = x^{**} \in (x)^\circ \subseteq I$, which is a contradiction. Hence I contains no dense elements.

(4) $\Rightarrow$ (5): Assume that every proper ideal contains no dense elements. Let P be a prime ideal. Then clearly $P \cap D = \emptyset$. Let $a \in P$ and $x \in (a)^\circ$. Hence $x \wedge a^* = 0 \in P$. Suppose $a^* \in P$. Then $a \vee a^* \in P \cap D = \emptyset$, which is a contradiction. Hence $a^* \notin P$, which implies that $x \in P$. Thus $(a)^\circ \subseteq P$. Therefore P is a normal ideal of L.

(5) $\Rightarrow$ (1): Assume that every prime ideal is normal. Let $x, y \in L$ such that $x^* = y^*$. Suppose $x \neq y$. Then there exists a prime ideal P such that $x \in P$ and $y \notin P$. Since P is normal and $x \in P$, we get $y \in P$, which is a contradiction. Hence L is disjunctive. $\square$

We now discuss some topological properties of prime normal ideals. Fir this, we first need the following results:

Theorem 2.15. *For any normal ideal I and a filter F of L such that $I \cap F = \emptyset$, there exists a prime normal ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Proof. Let I be a normal ideal and let F be a filter of L such that $I \cap F = \emptyset$. Consider $\sum = \{J \mid J \text{ is a normal ideal such that } I \subseteq J \text{ and } J \cap F = \emptyset\}$. Clearly $I \in \sum$. Clearly $\sum$ satisfies the hypothesis of Zorn's Lemma. Let M be a maximal element of $\sum$. Let $x, y \in L$ be such that $x \notin M$ and $y \notin M$. Then $M \subset M \vee (x] \subseteq (M \vee (x])^\circ$ and $M \subset M \vee (y] \subseteq (M \vee (y])^\circ$. By the maximality of, we get $(M \vee (x])^\circ \cap F \neq \emptyset$ and $(M \vee (y])^\circ \cap F \neq \emptyset$. Choose $a \in (M \vee (x])^\circ \cap F$ and $b \in (M \vee (y])^\circ \cap F$. Hence $a \wedge b \in F$ and

$$\begin{aligned} a \wedge b &\in (M \vee (x])^\circ \cap (M \vee (y])^\circ \\ &= (M \vee (x \wedge y])^\circ \end{aligned}$$

If $x \wedge y \in M$, then $a \wedge b \in M^\circ = M$. Hence $a \wedge b \in M \cap F$, which is a contradiction. Therefore M is prime. $\square$

Corollary 2.16. *Let I be a normal ideal of L and $x \notin I$. Then there exists a prime normal ideal P of L such that $I \subseteq P$ and $x \notin P$.*

Corollary 2.17. *For any normal ideal I of L, we have*

$$I = \bigcap \{P \mid P \text{ is a prime normal ideal of } L,\ I \subseteq P\}$$

Corollary 2.18. *The intersection of all prime normal ideals is $\{0\}$.*

Let L be a bounded distributive lattice and $Spec^{\circ}(L)$ denote the set of all prime normal ideals of L. For any $A \subseteq L$, let $K(A) = \{P \in Spec^{\circ}(L) \mid A \nsubseteq P\}$ and for any $x \in L, K(x) = K(\{x\})$. Then we have the following observations which can be verified directly.

Lemma 2.19. *For any $x, y \in L$, the following holds:*

(1) $K(x) \cap K(y) = K(x \wedge y)$
(2) $K(x) \cup K(y) = K(x \vee y)$
(3) $K(x) = \emptyset \Leftrightarrow x = 0$
(4) $K(1) = Spec^{\circ}(L)$

From the above lemma, it can be easily observed that the collection $\{K(x) | x \in L\}$ forms a base for a topology on $Spec^{\circ}(L)$ which is called a hull-kernel topology. Under this topology we have the following:

Theorem 2.20. *Let L be a pseudo-complemented distributive lattice. Then we have the following:*

(1) For any $x \in L$, $K(x)$ is compact in $Spec^{\circ}(L)$
(2) Let C be a compact open subset of $Spec^{\circ}(L)$. Then $C = K(x)$ for some $x \in L$
(3) $Spec^{\circ}(L)$ is a T_0-space.

Proof. (1). Let $x \in L$. Let $A \subseteq L$ be such that $K(x) \subseteq \bigcup_{y \in A} K(y)$. Let I be the ideal generated by the set A. Suppose $x \notin I^{\circ}$. Then by Corollary 2.16, there exits a prime normal ideal P such that $I^{\circ} \subseteq P$ and $x \notin P$. Hence $P \in K(x) \subseteq \bigcup_{y \in A} K(y)$. Therefore $y \notin P$ for some $y \in A$, which is a contradiction to that $I \subseteq I^{\circ} \subseteq P$. Therefore $x \in I^{\circ}$. Then we get $x \in (a)^{\circ}$ for some $a \in I$. Since I is the ideal generated by A, we get $a \leq a_1 \vee a_2 \vee ... \vee a_n$ for some $a_1, a_2, ..., a_n \in A$. Hence $x \in (a)^{\circ} \subseteq (a_1 \wedge a_2 \wedge ... \wedge a_n)^{\circ}$. Then clearly $K(x) \subseteq \bigcup_{i=1}^{n} K(a_i)$, which is a finite subcover of $K(x)$. Therefore for each $x \in L$, $K(x)$ is a compact open subset of $Spec^{\circ}(L)$.

(2). Let C be a compact open subset of $Spec^{\circ}(L)$. Since C is open, we get $C = \bigcup_{a\in A} K(a)$ for some $A \subseteq L$. Since C is compact, there exists $a_1, a_2, ..., a_n \in A$ such that

$$C = \bigcup_{i=1}^{n} K(a_i) = K(\bigvee_{i=1}^{n} a_i)$$

Therefore $C = K(x)$ for some $x \in L$.

(3). Let P, Q be two distinct prime normal ideals of L. Without loss of generality assume that $P \nsubseteq Q$. Choose $x \in L$ such that $x \in P$ and $x \notin Q$. Hence $P \notin K(x)$ and $Q \in K(x)$. Therefore $Spec^{\circ}(L)$ is a T_0-space. □

In [10], T.P. Speed exclusively studied the topological properties of the space $Minp\ (L)$ of all minimal prime ideals of distributive lattices. We have already observed that every minimal prime ideal is a normal ideal. Now we derive some equivalent conditions for every prime normal ideal of a lattice to become a minimal prime ideal.

Theorem 2.21. *Let L be a bounded distributive lattice. Then the following conditions are equivalent:*

(1) Every prime normal ideal is a minimal prime ideal

(2) $Spec^{\circ}(L) = Minp\ (L)$

(3) Each $K(x)$ is closed in $Spec^{\circ}(L)$

(4) $Spec^{\circ}(L)$ is Hausdorff

(5) For any $x, y \in L$, there exists $z \in L$ such that $x \wedge z = 0$ and

$$K(y) \cap \{Spec^{\circ}(L) - K(x)\} = K(z \wedge x^*)$$

Proof. (1) ⇒ (2): It is clear.

(2) ⇒ (3): Assume the condition (2). Then each $K(x) = \mu(x)$ is closed in $Minp\ (L) = Spec^{\circ}(L)$.

(3) ⇒ (4): Assume the condition (3). Let P, Q be two distinct prime normal deals of L. Choose $x \in P$ and $x \notin Q$. Then $Q \in K(x)$ and $P \in Spec^{\circ}(L) - K(x)$. Since $K(x)$ is closed, we get that $Spec^{\circ}(L) - K(x)$ is open and hence there exists $y \in L$ such that $P \in K(y) \subseteq Spec^{\circ}(L) - K(x)$. It is also clear that $K(x) \cap K(y) = \emptyset$. Therefore $Spec^{\circ}_F(L)$ is a Hausdorff space.

(4) $\Rightarrow$ (5): Assume that $Spec^{\circ}(L)$ is a Hausdorff space. Hence $K(a)$ is a compact subset of $Spec^{\circ}(L)$, for each $a \in L$. Then $K(a)$ is a clopen subset of $Spec^{\circ}(L)$. Let $x, y \in L$ such that $x \neq y$. Then $K(y) \cap \{Spec^{\circ}(L) - K(x)\}$ is a compact subset of the compact space $K(y)$. Since $K(y)$ is open in $Spec^{\circ}(L)$, $K(y) \cap \{Spec^{\circ}_{F}(L) - K(x)\}$ is a compact open subset of $Spec^{\circ}(L)$. Hence by Theorem 2.18(2), there exists $z \in L$ such that

$$K(z) = K(y) \cap \{Spec^{\circ}(L) - K(x)\}$$

Also $K(x \wedge z) = K(x) \cap K(z) = \emptyset$. Therefore $x \wedge z = 0$. Therefore $x^{*} \wedge z = z$. Therefore $K(y) \cap \{Spec^{\circ}(L) - K(x)\} = K(z) = K(z \wedge x^{*})$.

(5) $\Rightarrow$ (1): Let P be a prime normal ideal of L. Choose $x, y \in L$ such that $x \in P$ and $y \notin P$. Then by the condition (5), there exists $z \in L$ such that $x \wedge z = 0$ and

$$K(y) \cap \{Spec^{\circ}(L) - K(x)\} = K(x^{*} \wedge z)$$

Then clearly $P \in K(y) \cap \{Spec^{\circ}(L) - K(x)\} = K(x^{*} \wedge z)$. If $z \in P$, then $x^{*} \wedge z \in P$, which is a contradiction to that $P \in K(x^{*} \wedge z)$. Hence $z \notin P$. Thus for each $x \in P$, there exists $z \notin P$ such that $x \wedge z = 0$. Therefore P is a minimal prime ideal of L. □

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