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A New Quarter Sweep Arithmetic Mean (QSAM) Method to Solve Diffusion Equations

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Abstract: The aim of this paper is to introduce the Quarter-Sweep Arithmetic Mean (QSAM) method using the Quarter-Sweep Crank-Nicolson (QSCN) finite difference method for solving one-dimensional diffusion equations. The formulation of the QSAM method is developed by combining the concept of the quarter-sweep iteration and the Arithmetic Mean (AM) method known as one of two-step iterative methods. The QSAM method has been shown to be very fast as compared to the standard AM method. Some numerical tests were included to support our statement.

Keywords and Phrases: Quarter-Sweep Iteration, Arithmetic Mean Algorithm, QSCN Finite Difference

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1 Introduction

By considering numerical techniques, there are many methods can be used by researchers to gain approximate solutions such as the finite difference, finite element, finite volume and boundary element methods. In solving any partial differential

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equation, however, those methods will discretize the equations to the approximation equation, which is forming a system of linear equations.

Consequently, the concept of the two-stage iterative methods is definitely one of the efficient methods in solving any system of linear equations. Actually, there are many two-stage iterative methods have been proposed such as the Alternating Group Explicit (AGE) [7], the Iterative Alternating Decomposition Explicit (IADE) [10], the Reduced Iterative Alternating Decomposition Explicit (RIADE) [11], the Half-Sweep Iterative Alternating Decomposition Explicit (HSIADE) [12], the Quarter-Sweep Iterative Alternating Decomposition Explicit (QSIADE) [13], the Arithmetic Mean (AM) [9], and Block Jacobi [2] methods. However, this standard AM method can be also named as the Full-Sweep Arithmetic Mean (FSAM) method. In 2004, the FSAM method has been modified [14] by borrowing the concept of the half-sweep iteration [1] and then called as the Half-Sweep Arithmetic Mean (HSAM) method. Further studies of the HSAM method have been conducted by Sulaiman *et al.* ([15], [16]). Similarly by applying the quarter-sweep iteration, we, here, introduce a new AM scheme called the Quarter-Sweep Arithmetic Mean (QSAM) method.

To investigate the effectiveness of the QSAM method, let us consider the one-dimensional diffusion equation as given by

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2}, \quad a \le x \le b, \ 0 \le t \le T, \tag{1}$$

subject to the initial condition

$$U(x,t) = g_1(x), \quad a \le x \le b,$$

and the boundary conditions

$$\left.\begin{array}{ll} U(a,t) = & g_2(t) \\ U(b,t) = & g_3(t) \end{array}\right\} \quad 0 \le t \le T,$$

where α is a diffusion parameter.

Before describing formulation of the finite difference approximation equation in case of the full-, half-, and quarter-sweep iterations over the problem (1), we assume the solution domain (1) can be uniformly divided into $(n+1)$ and M subintervals in the x and t directions. The subintervals in the x and t directions are denoted Δx and Δt respectively, which are uniform and defined as

$$\left.\begin{array}{ll} \triangle x = & \frac{b-a}{m} = h, m = n+1 \\ \triangle t = & \frac{T-0}{M} \end{array}\right\}. \tag{2}$$

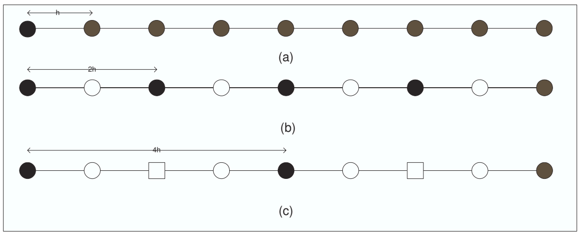

Figure 1: a)., b).and c). show the distribution of uniform node points for the full-, half-, and quarter-sweep cases respectively.

2 Formulation of Quarter-Sweep Finite Difference Approximation

Based on Fig. 1, we firstly need to build three finite grid networks in order to derive the full-, half-, and quarter-sweep finite difference approximation equations by discretizing the problem (1). Actually, these networks show us on implementations of the full-, half-, and quarter-sweep iterative algorithms applied onto the node points of type $\bullet$ only until iterative convergence test is achieved. Then other approximate solutions at remaining points (points of the different type) are computed directly using the direct method, see [1], [8], [17].

Because of implementations of the full-, half-, and quarter-sweep iterations involve the node points of type $\bullet$ only, it is obvious that the implementation of the half- and quarter-sweep iterative methods just involves approximately 50% and 25% of the whole inner points as shown in Figs. 1b and 1c compared to the full-sweep iterative method. In this paper, the central difference and Crank-Nicolson (CN) approaches have been used to derive the Full-, Half- and Quarter-Sweep Crank-Nicolson finite difference approximation equations, which are indicated as the FSCN, HSCN, and QSCN respectively. All CN finite difference approximation equations at the $(j+1)$ time level, can generally be expressed as

$$-\beta_1 U_{i-p,j+1} + \beta_2 U_{i,j+1} - \beta_1 U_{i+p,j+1} = f_{i,j}^{\beta}, \tag{3}$$

where

$$\beta_1 = \frac{\alpha \triangle t}{2(ph)^2}, \beta_2 = \left(1 + \frac{\alpha \triangle t}{(ph)^2}\right), \beta_3 = \left(1 - \frac{\alpha \triangle t}{(ph)^2}\right),$$

$$f_{i,j}^{\beta} = \beta_1 U_{i-p,j} + \beta_3 U_{i,j} + \beta_1 U_{i+p,j}.$$

The values of p, which correspond to 1, 2 and 4, represent the full-, half- and quarter-sweep cases respectively. Eq. (3), considered at the $(j+1)$ time level, generates a system of linear equation as follows

$$A_\beta \underline{U}^\beta_{j+1} = \underline{f}^\beta_j, \tag{4}$$

where coefficient matrix, A_β is given by

$$A_\beta = \begin{bmatrix} \beta_2 & -\beta_1 & & & & \\ -\beta_1 & \beta_2 & -\beta_1 & & & \\ & -\beta_1 & \beta_2 & -\beta_1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\beta_1 & \beta_2 & -\beta_1 \\ & & & & -\beta_1 & \beta_2 \end{bmatrix}_{(\frac{m}{p}-p)\times(\frac{m}{p}-p)} .$$

3 Quarter-Sweep Arithmetic Mean Methods

For simplifying, let Eq. (4) at any time level be rewritten as

$$A\underline{U} = \underline{f}, \tag{5}$$

where,

$$A = \begin{bmatrix} a & b & & & & \\ c & a & b & & & \\ & c & a & b & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & a & b \\ & & & & c & a \end{bmatrix}_{(\frac{m}{p}-1)\times(\frac{m}{p}-1)} .$$

In the next discussion, we show on how to present the formulation of the FSAM, HSAM, and QSAM schemes. As stated in previous section, all AM methods are one of two-stage iterative methods. It denotes that the iterative process for all methods involve two levels of virtual time such as $\underline{U}^{(1)}$ and $\underline{U}^{(2)}$. To develop formulation of all AM methods in Eq. (5), suppose the symmetry coefficient matrix, A (5) needs to be decomposed into

$$A = L + D + T, \tag{6}$$

where L, D and T are strictly lower triangular, diagonal and strictly upper triangular matrices respectively. The general scheme for all AM methods is given by

$$\left.\begin{array}{l}(D + rL)\underline{U}^{(1)} = ((1-r)D - rT)\underline{U}^{(k)} + rf \\ (D + rT)\underline{U}^{(2)} = ((1-r)D - rL)\underline{U}^{(k)} + rf \\ \underline{U}^{(k+1)} = \frac{\underline{U}^{(1)}+\underline{U}^{(2)}}{2}\end{array}\right\}. \quad (7)$$

where r and $\underline{U}^{(k)}$ represent as an acceleration parameter and an unknown vector at the kth iteration respectively. Practically the value of r will be determined by implementing some computer programs and then choose one value of r, where its number of iterations is the smallest. By determining values of matrices L, D and T as stated in Eq. (6), the general algorithm for all AM schemes in Eq. (7) may be described in Algorithm 1. Generally the basic idea for the convergence analysis of the AM methods has been proved by [9]. The FSAM, HSAM, and QSAM algorithms are explicitly performed by using all equations at level (1) and level (2) alternatively until the specified convergence test is satisfied. The FSAM method, however, will be used as the control of comparison of numerical results.

4 Numerical Results

To verify the efficiency of the implementation of the QSAM scheme as derived in Eq. (5), which is based on the approximation Eq. (3), there are three items will be considered in comparison such as the number of iterations, execution time and maximum absolute error. Some numerical experiments were conducted in solving the following one-dimensional diffusion equation as follows

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \quad 0 \le x \le 1, \ 0 \le t \le 1.0. \quad (8)$$

The initial and boundary conditions and exact solution of the problem (8) are given by

$$U(x,t) = e^{-(\pi^2 t)} \sin(\pi x), \quad 0 \le x \le 1, \ 0 \le t \le 1.0. \quad (9)$$

Algorithm I: FSAM, HSAM and QSAM schemes

Phase I. Preprocessing step

1.1 Choose the optimal parameter, r.

1.2 Set initial vector $U_i^{(0)} = 0.0, \qquad i = 1p, 2p, 3p, \ldots, m-p,$

Phase II. AM iteration

A) at level (1)

2.1. Set $a \leftarrow \beta_2, \quad b \leftarrow -\beta_1, \quad c \leftarrow -\beta_1,$

2.2. Set $w \leftarrow a(1-r), \quad v \leftarrow rb, \quad \lambda \leftarrow rc,$

2.3. For $i = 1p, 2p, 3p, \ldots, m-p,$

$$\text{Calculate } U_i^{(1)} \leftarrow \begin{cases} (wU_i^{(k)} - vU_{i+1}^{(k)} + rf_i)/a & i = 1p \\ (wU_i^{(k)} - \lambda U_{i-1}^{(1)} + rf_i)/a & i = m-p \\ (wU_i^{(k)} - vU_{i+1}^{(k)} - \lambda U_{i-1}^{(1)} + rf_i)/a & others \end{cases}$$

B) at level (2)

2.4. For $i = m-p, m-2p, \ldots, 1,$

$$\text{Calculate } U_i^{(2)} \leftarrow \begin{cases} (wU_i^{(k)} - vU_{i+1}^{(2)} + rf_i)/a & i = 1p \\ (wU_i^{(k)} - \lambda U_{i-1}^{(k)} + rf_i)/a & i = m-p \\ (wU_i^{(k)} - vU_{i+1}^{(2)} - \lambda U_{i-1}^{(k)} + rf_i)/a & others \end{cases}$$

2.5. For $i = 1p, 2p, 3p, \ldots, m-p,$

Calculate $U_i^{(k+1)} \leftarrow \frac{1}{2}(U_i^{(1)} + U_i^{(2)})$

All results of numerical experiments, which were gained from implementations of the FSAM, HSAM, and QSAM methods, have been recorded in Table 1. In implementations mentioned above, the convergence test considered the tolerance error $\varepsilon = 10^{-10}$. Figs. 2 and 3 show number of iterations and execution time against mesh size respectively.

5 Conclusion

As mentioned in the second section, the formulation of the FSCN, HSCN and QSCN approximation equations, based on the CN scheme, can be easily formulated and rewritten in the general form as shown in Eq. (3). Through the observation in Table 1 and it has shown in Fig. 2 that number of iterations decreased approximately 65.79-70.54% and 42.08-45.74% respectively correspond to the QSAM and HSAM methods compared to the FSAM method. In fact, the execution time against the mesh size of both the QSAM and HSAM methods are much faster

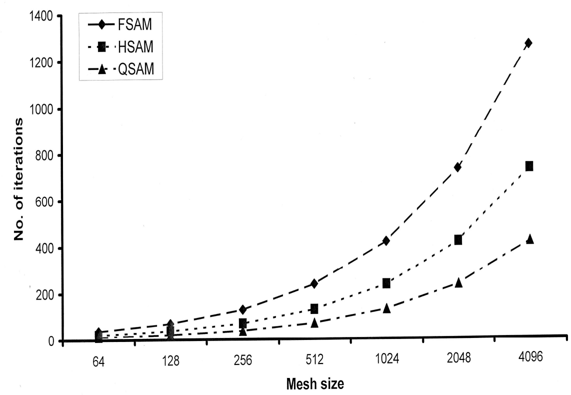

Figure 2: Number of iterations versus mesh size of the FSAM, HSAM, and QSAM methods.

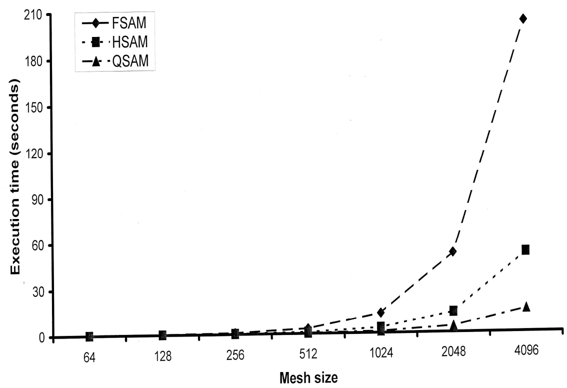

Figure 3: The execution time (seconds) versus mesh size of the FSAM, HSAM, and QSAM methods.

Table 1: Comparison of number of iterations K, the execution time (seconds) and maximum errors for the iterative methods ($\alpha = 1.0; \Delta t = 0.01$) with optimal value r.

M	FSAM				HSAM				QSAM			
	r	**K**	**Time**	**Error**	**r**	**K**	**Time**	**Error**	**r**	**K**	**Time**	**Error**
64	1.75	38	0.06	3.1293e-7	1.55	22	0.02	4.6097e-9	1.32	13	0	1.2524e-6
128	1.87	70	0.29	3.9151e-7	1.75	38	0.07	3.1293e-7	1.55	22	0.02	3.0272e-8
256	1.934	129	0.98	4.1496e-7	1.87	70	0.38	3.9151e-7	1.75	38	0.07	3.1294e-7
512	1.9675	237	3.36	4.3019e-7	1.934	129	1.04	4.1496e-7	1.87	70	0.39	3.9151e-7
1024	1.98461	418	13.11	4.3975e-7	1.9675	235	3.51	4.2738e-7	1.934	129	1.08	4.1496e-7
2048	1.99299	731	51.77	4.6089e-7	1.98461	418	13.36	4.3975e-7	1.9675	235	4.08	4.2738e-7
4096	1.99685	1262	202.15	5.1353e-7	1.99299	731	52	4.6089e-7	1.98461	418	15.31	4.3975e-7

about 92.46-100.00% and 50.00-96.84% respectively than the FSAM method, see Fig. 3.

Overall, the numerical results have proven that the QSAM method is more superior in terms of number of iterations and the execution time than the FSAM or HSAM method. This is due to the computational complexity of the QSAM method is approximately 75% less than of the FSAM method.

For future works, the further investigation for the capability of family of AM methods need to be carried out for solving other multi-dimensional partial differential equations ([3], [7], [5]) together with or without forcing term [6]. In fact, further studies should also be conducted for comparison ammong two-stage iterative methods such as AGE [7], IADE ([10], [11], [12], [13]), AM [9] and TSAM [4], and Block Jacobi [2].

References

[1] A.R. Abdullah, The Four Point Explicit Decoupled Group (EDG) Method: A Fast Poisson Solver, *Int. J. Computer Maths.*, **38**(1991), 61–70.

[2] T. Allahviranloo, E. Ahmady, N. Ahmady and K.S. Alketaby, Block Jacobi two-stage method with Gauss-Sidel inner iterations for fuzzy system of linear equations, *Applied Mathematics and Computation*, **175**(2006), 1217–1228.

[3] D.J. Evans, Group Explicit Iterative methods for solving large linear systems, *Int. J. Computer Maths.*, **17**(1985), 81–108.

[4] E. Galligani and V. Ruggiero, The Two-Stage Arithmetic Mean Method, *Applied Mathematics and Computation*, **85**(1997), 245–264.

[5] A. Ibrahim and A.R. Abdullah, Solving the two dimensional diffusion equation by the Four Point Explicit Decoupled Group (EDG) iterative method, *Int. J. Computer Maths.*, **58**(1995), 253–256.

[6] Y. Lin and C. Xu, Finite difference/spectral approximations for the time-fractional diffusion equation., *Journal of Computational Physics*, **255**(2007), 1533–1552.

[7] D.J. Evans and M.S. Sahimi, The Alternating Group Explicit iterative method (AGE) to solve parabolic and hyperbolic partial differential equations, *Ann. Rev. Num. Fluid Mechanic and Heat Trans*, **2**(1988), 283–389.

[8] A. Ibrahim, The Study of the Iterative Solution Of Boundary Value Problem by the Finite Difference Methods, PhD Thesis, Universiti Kebangsaan Malaysia, (1993).

[9] V. Ruggiero and E. Galligani, An iterative method for large sparse systems on a vector computer, *Computer Math. Applic.*, **20**(1990), 25–28.

[10] M.S. Sahimi, A. Ahmad and A.A. Bakar, The Iterative Alternating Decomposition Explicit (IADE) method to solve the heat conduction equation, *Int. J. Computer Maths.*, **47**(1993), 219–229.

[11] M.S. Sahimi and M. Khatim, The Reduced Iterative Alternating Decomposition Explicit (RIADE) Method for the Diffusion Equation, *Pertanika J. Sci. and Technol.*, **9(1)**(2001), 13–20.

[12] J. Sulaiman, M.K. Hasan and M. Othman, *The Half-Sweep Iterative Alternating Decomposition Explicit (HSIADE) method for diffusion equations*, Lecture Note on Computer Science (LNCS 3314), Springer-Verlag, Berlin (2004), 57–63.

[13] J. Sulaiman, M. Othman and M.K. Hasan, Quarter-Sweep Iterative Alternating Decomposition Explicit algorithm applied to diffusion equations, *Int. J. Computer Maths.*, **81(12)**(2004), 1559–1565.

[14] J. Sulaiman, M. Othman and M.K. Hasan, A new Half-Sweep Arithmetic Mean (HSAM) algorithm for two-point boundary value problems, *Proceedings of the International Conference on Statistics and Mathematics and Its Application in the Development of Science and Technology*, (2004), 169–173.

[15] J. Sulaiman, M. Othman and M.K. Hasan, A fourth-order finite difference solver for Poisson's equations via the Half-Sweep Arithmetic Mean (HSAM) method, *Proceedings of the First IMT-GT Regional Conference Mathematics, Statistics and Their Applications*, (2005), 139–146.

[16] J. Sulaiman, M. Othman and M.K. Hasan, Half-sweep Arithmetic Mean method using finite element approximation for Poisson's equation, *Proceedings of the International Conference on Applied Mathematics*, (2005), In CD Form. (ISBN: 90-365-2244-7).

[17] W.S. Yousif and D.J. Evans, Explicit De-coupled Group iterative methods and their implementations, *Parallel Algorithms and Applications*, **7**(1995), 53–71.

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