Chamchuri Journal of Mathematics
VOLUME 4(2012), 71–77
http://www.math.sc.chula.ac.th/cjm

Matrix Transformations into A New Sequence Space Related to Invariant Means

Ab Hamid Ganie* and Neyaz Ahmad Sheikh

Received 15 May 2012
Accepted 26 December 2012

Abstract:

In this paper we introduce the sequence space $V_\infty(\theta)$ through the concept of invariant means and lacunary sequence space $\theta = (k_r)$, show its completeness property and characterize the matrix classes $(l_\infty : V_\infty(\theta))$ and $(l(p) : V_\infty(\theta))$.

Keywords: Lacunary sequence, invariant means, almost lacunary convergence, matrix transformation

2000 Mathematics Subject Classification: 46A45, 40C05

1 Preliminaries, Background and Notations

A sequence space is defined to be a linear space of real or complex sequences. Throughout the paper $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the set of non-negative integers, the set of real numbers and the set of complex numbers respectively. Let ω denote the space of all sequences (real or complex); l_∞, c and c respectively, denotes the space of all bounded sequences, the space of convergent sequences and null sequences. Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup_k p_k = H$ and $M = max\{1, H\}$. Then, the linear

* *Corresponding author*

spaces $l(p)$ and $l_\infty(p)$ were defined by Maddox [8] (see, also [9, 13, 18, 19]) as follows :

$$l(p) = \{x = (x_k) : \sum_k |x_k|^{p_k} < \infty\}$$

and

$$l_\infty(p) = \{x = (x_k) : \sup_k |x_k|^{p_k} < \infty\},$$

which are complete spaces paranormed by

$$g_1(x) = \left[\sum_k |x_k|^{p_k}\right]^{1/M} \quad \text{and} \quad g_2(x) = \sup_k |x_k|^{p_k/M} \ \text{ iff } \ \inf p_k > 0.$$

We shall assume throughout that $p_k^{-1} + (p_k{}')^{-1}$ provided $1 < \inf p_k \leq H < \infty$.

Let X and Y be two sebsets of ω. Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, the matrix A defines the $A-$transformation from X into Y, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x exists and is in Y; where $(Ax)_n = \sum_k a_{nk}x_k$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(X : Y)$, we denote the class of all such matrices. A sequence x is said to be A-summable to l if Ax converges to l which is called as the A-limit of x (see [2, 7, 8, 15, 17]).

Let σ be a one-to-one mapping from the set of positive integers into itself. A continuous linear functional ϕ on l_∞ is said to be an invariant mean (or a σ mean) if and only if

(i) $\phi(x) \geq 0$,when the sequence $x = (x_n)$ has $x_n \geq 0$, for all n,

(ii) $\phi(e) = 0$, where $e = (1, 1, \ldots)$,

and

(iii) $\phi(x_{\sigma(n)} = \phi(x)$ for all $x \in l_\infty$.

If σ is the translation mapping $n \to n + 1$, then a σ-mean is often called a Banach limit(see, [4, 6, 16]). A σ-mean extends the limit functional on c in the sense that $\phi(x) = \lim\ x$, for all $x \in c$, if and only if σ has no finite orbits, that is, if and only if for all $n \geq 0, j \geq 1, \sigma^j(n) \neq n$ (see, [11, 12]).

A sequence $x \in l_\infty$ is said to be a σ-convergent sequence if all its σ-means (or invariant means) are equal. We denote the set of all σ-convergent sequences by V_σ. If $x = (x_n)$, we write $Tx = (Tx_n) = (x_{\sigma(n)})$, then (see [14])

$$V_\sigma = \left\{ x \in l_\infty : \lim_{p\to\infty} t_{pn}(x) = L, \text{ uniformly in } n, \ L = \sigma - \lim \ x \right\},$$

where,

$$t_{pn}(x) = \frac{1}{p+1} \sum_{m=0}^{p} x_{\sigma^m(n)}.$$

By lacunary sequence space we mean an increasing seuence $\theta = (k_r)$ of integers, such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$. Throughout the text the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r (see [1, 3, 11, 12]).

2 σ-lacunary bounded sequences

In this section we define the space $V_\infty(\theta)$ and investigate its completeness property.

Following [2, 7, 8, 13, 16], we define the space $V_\infty(\theta)$ as follows:

$$V_\infty(\theta) = \left\{ x \in l_\infty : \sup_{r,n} |t_{rn}(x)| < \infty \right\},$$

where,

$$t_{rn}(x) = \frac{1}{h_r} \sum_{j\in I_r} x_{\sigma^j(n)}.$$

We call the space $V_\infty(\theta)$ as the space of σ-lacunary bounded sequences. It is clear that $c \subset V_\sigma \subset V_\infty(\theta) \subset l_\infty$.

Theorem 2.1. *The space $V_\infty(\theta)$ is a Banach space normed by*

$$||x|| = \sup_{r,n} |t_{rn}(x)|. \tag{2.1}$$

Proof. The proof is a routine verification, so we left an easy exercise for the reader. □

3 Matrix Transformation in $V_\infty(\theta)$

In this section we characterize the matrix classes $(l_\infty(p) : V_\infty(\theta))$ and $(l(p) : V_\infty(\theta))$.

For the sake of brevity in notation, we shall write,

$$t_{rn}(Ax) = \sum_{k=1}^{\infty} t(n,k,r)x_k \tag{3.1}$$

where, $t(n,k,m) = \frac{1}{h_r} \sum_{j\in I_r} a(\sigma^j(n),k)$ and $a(n,k)$ denotes the elements a_{nk} of the infinite matrix A.

Theorem 3.1. *$A \in (l_\infty(p) : V_\infty(\theta))$ if and only if there exists $M > 1$ such that* $\sup_{n,r} \sum_k |t(n,k,r)|\, M^{\frac{1}{p_k}} < \infty$.

Proof. **Sufficiency**: Suppose that $x = (x_k) \in l_\infty(p)$. We have

$$\begin{aligned} |t_{rn}(Ax)|^{p_k} &\leq \left|\sum_k t(n,k,r)\right|^{p_k} |x|^{p_k} \\ &\leq \left|\sum_k t(n,k,r)\right|^{p_k} \sup_k |x|^{p_k}. \end{aligned}$$

Now taking supremum over n, r on both sides, we get $Ax \in V_\infty(\theta)$ i.e., $A \in (l_\infty(p) : V_\infty(\theta))$.

Necessity: Let $A \in (l_\infty(p) : V_\infty(\theta))$. Let us write $|q_{rn}(x)| = \sup_r |t_{rn}A(x)|$. Then it is easy to that for each $n \geq 0$, $q_{rn}(x)$ is a continuous semi norm on $l_\infty(p)$ and $q_{rn}(x)$ is point wise bounded on $l_\infty(p)$. Assume to the contrary that (3.1) is not true. Then there exists $x = (x_k) \in l_\infty(p)$ with $\sup_n q_{rn}(x) = \infty$. By the principal of condensaton of singularities [11], the set

$$\left\{ x = (x_k) \in l_\infty(p) : \sup_n q_{rn}(x) M^{\frac{1}{p_k}} = \infty \right\},$$

is of second category in $l_\infty(p)$ and hence non-empty, that is $x = (x_k) \in l_\infty(p)$ with $\sup_n q_{rn}(x) = \infty$. But this is contradict to to the fact that q_n is point wise bounded on $l_\infty(p)$. Therefore, by Banach-Steinhauss theorem, there is H such that

$$q_{rn}(x) \leq H\|x\|_1. \tag{3.2}$$

Now define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} sgn\ t(n,k,r), & \text{for each } n, r, 1 \leq k \leq k_0, \\ 0, & \text{for } k > k_0. \end{cases}$$

Then $x \in l_\infty(p)$. Applying this sequence to (3.2) we get (3.1), that completes the proof. □

Theorem 3.2. *$A \in (l(p) : V_\infty(\theta))$ if and only if*

$$\sup_{n,r} \sum_k |t(n,k,r)|^{q_k} B^{-q_k} < \infty \qquad (1 < p_k < \infty) \tag{3.3}$$

and

$$\sup_{n,k,r} |t(n,k,r)|^{p_k} < \infty \qquad (0 < p_k \leq 1).$$

Proof. We only consider the case $1 < p_k < \infty$, for every $k \in \mathbb{N}$ and the case $0 < p_k \leq 1$ will follow similarly.

Sufficiency: Suppose that (3.3) holds and $x \in l(p)$. Now, for any $C > 0$ and any two complex numbers a and b,

$$|ab| \leq C(\left|aC^{-1}\right|^q + |b|^p) \qquad \text{(see [6])}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

We have for some integer $B > 1$ that

$$\sum_k |t(n,k,r)x_k| \leq B\left(\sum_k |t(n,k,r)x_k|^{q_k} B^{-q_k} + |x_k|^{p_k}\right)$$

for every $x \in l(p)$. Taking the supremum over n, r on both sides and using (3.3), we get $Ax \in V_\infty(\theta)$ for every $x \in l(p)$ i.e., $A \in (l(p) : V_\infty(\theta))$.

Necessity: Let $A \in (l(p) : V_\infty(\theta))$ and $x \in l(p)$. We put

$$p_n(x) = \sup_r \sum_k |t(n,k,r)|^{p_k}.$$

Then it is easy to see that for each $n \geq 0$, $p_n(x)$ is a continuous semi norm on $l(p)$ and $p_n(x)$ is point wise bounded on $l(p)$. assume to the contrary that (3.3) is not true. Then there exists $x \in l(p)$ with $\sup_n p_n(x) = \infty$. by the principal of condensation of singularities (see [11]), the set

$$\left\{ x \in l(p) : \sup_n p_n(x) = \infty \right\},$$

is of second category in $l(p)$ and hence non-empty, that is $x \in l(p)$ with $\sup_n p_n(x) = \infty$. But this is contradiction to the fact that $p_(n)$ is bounded on $l(P)$. Thus, by Banach-Steinhauss theorem, there is a constant $M >)$, such that

$$p_n(x) \leq M\|x\|. \tag{3.4}$$

Now, applying (3.4) to the sequence $x = (x_k)$ defined by Lascarides (see [5]) by replacing a_{nk} by $t(n, k, r)$, we obtained the necessity of (3.3). This completes the proof. □

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Ab Hamid Ganie
Department of Mathematics
National Institute of Technology
Srinagar, INDIA-190006
Email: ashamidg@rediffmail.com

Neyaz Ahmad Sheikh
Department of Mathematics
National Institute of Technology
Srinagar, INDIA-190006
Email: neyaznit@yahoo.co.in