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Survey Article

The forest number in several classes of regular graphs

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Abstract: For a graph G and $F \subseteq V(G)$, if $\langle F \rangle$ is acyclic, then F is said to be an induced forest of G. The size of a maximum induced forest of G is called the forest number of G and is denoted by $\mathsf{f}(G)$. The purpose of this paper is to provide a review of recent results and open problems on the forest number of graphs and in some classes of graphs.

Keywords: forest number, path number, tree number

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1 Introduction

We limit our discussion to graphs that are simple and finite. For the most part, our notation and terminology follows that of Chartrand and Lesniak [9]. Let $G = (V, E)$ be a graph. It is well known that the *cycle rank* or *Betti number* of G is the minimum number of edges that must be removed in order to eliminate all cycles in G. The cycle rank of G is denoted by $\mathsf{b}(G)$. It is also well known that $\mathsf{b}(G)$ has a simple expression, namely, $\mathsf{b}(G) = |E(G)| - |V(G)| + c(G)$, where $c(G)$ denotes the number of components of G. The corresponding problem of eliminating all cycles from a graph by means of deletion of vertices does not have a simple solution.

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For a graph G and $F \subseteq V(G)$, if an induced subgraph $\langle F \rangle$ is acyclic, then F is said to be an induced forest of G. The size of a maximum induced forest of G is called the forest number of G and is denoted by $\mathsf{f}(G)$. The *path number* and the *tree number* of a graph G can be defined as the maximum order of an induced path of G denoted by $\mathsf{p}(G)$ and the maximum order of an induced tree of G denoted by $\mathsf{t}(G)$. The graph parameter $\mathsf{t}(G)$ was first introduced by Erdős, Saks and Sós [10]. The following theorems were proved in [10].

Theorem 1.1. *Let G be a connected graph of order n and size m. Then*

$$\mathsf{t}(G) \geq \frac{2n}{m-n+3}.$$

Theorem 1.2. *Let G be a connected graph of order n and radius r. Then*

$$\mathsf{p}(G) \geq 2r - 1.$$

There is a fairly large literature of papers dealing with the forest number of a graph, dating from the 1980's through the present. See, for example, [3], [6], and [31]. Contemporary interest in this invariant has significantly remained, in part due to its complementary invariant, known as the *decycling number*, denoted by $\phi(G)$, see [4] and [5]. The decycling number of a graph G is the minimum number of vertices that must be removed from G in order to obtain an acyclic graph. Thus determining the decycling number of a graph is equivalent to finding the maximum order of an induced forest and the sum of the two numbers equals the order of the graph. The problem of determining the minimum number of vertices of G whose removal eliminates all cycles in a graph G is known as the *decycling number* of G, and is denoted by $\phi(G)$. Thus for a graph G of order n, $\phi(G) + \mathsf{f}(G) = n$. The decycling number was first proposed by Beineke and Vandell [5].

Note that for a graph G, it is easy to see that F is a maximum induced forest of G if and only if $S = V(G) - F$ has the minimum cardinality among all subsets of $V(G)$ whose removal eliminates all cycles in G.

It was shown in [14], that determining the decycling number of an arbitrary graph is NP-complete (see Problem 7 on the feedback node set in the main theorem of [14], which asks for a set $S \subseteq V(G)$ of minimum cardinality in a digraph G such that every directed cycle of G contains a member of S. In fact, the computation of decycling numbers of the following families of graphs is shown to be NP-hard, namely, planar graphs, bipartite graphs, perfect graphs, and comparability graphs (graphs with a transitive orientation). On the other hand, the problem is known to be polynomial for various other families, including cubic graphs (see [16, 29]), permutation graphs (see Liang [17]), and interval and comparability graphs (see Liang and Chang [18]). These results naturally suggest further investigations as some good bounds on the parameter and exact results when possible. The followings are examples of graphs in which their forest numbers are easily obtained.

1. Let G be a graph of order n. Then $\mathsf{f}(G) = n$ if and only if G is a forest.
2. Let G be a graph of order n. Then $\mathsf{f}(G) = n - 1$ if and only if G has at least one cycle and there is a vertex on all of its cycles.
3. Let G be a graph of order $n \geq 2$. Then $\mathsf{f}(G) = 2$ if and only if $G \cong K_n$.
4. $\mathsf{f}(K_{r,s}) = 1 + s$ if r and s are positive integers satisfying $1 \leq r \leq s$.
5. $\mathsf{f}(K_{r_1,r_2 \cdots r_k}) = 1 + r_k$ if $r_1, r_2, \ldots, r_k$ are positive integers satisfying $1 \leq r_1 \leq r_2 \leq \cdots \leq r_k$.
6. For the Petersen graph P, $\mathsf{f}(P) = 7$.

A review of recent results and open problems on the decycling number is provided by Bau and Beineke [4].

2 Hypercubes and some other families of graphs

The n-dimensional cube (or n-cube) Q_n can be defined recursively: $Q_1 = K_2$ and $Q_n = K_2 \times Q_{n-1}$. An equivalent formulation, as the graph having $\mathbb{Z}_2^n$ as its vertex set with two vertices adjacent if they differ by exactly one co-ordinate. The following results of [5] give a lower bound for $\phi(Q_n)$ and it is equivalent to an upper bound for $\mathsf{f}(Q_n)$ as we will state in the followings.

1. Let $n \geq 2$, then $\mathsf{f}(Q_n) \leq 2\mathsf{f}(Q_{n-1})$.
2. Let $n \geq 2$, then $\mathsf{f}(Q_n) \leq 2^{n-1} + \frac{2^{n-1}-1}{n-1}$.
3. For $n \leq 8$, $\mathsf{f}(Q_n)$ was determined in [5].

n	1	2	3	4	5	6	7	8
$\mathsf{f}(Q_n)$	2	3	5	10	18	36	72	144

While the problem of determining $\mathsf{f}(Q_n)$ has not yet been solved, Pike proved some results in [19] as follows. Let $A(n,4)$ denote the size of a maximum binary code of length n and minimum Hamming distance 4. Then $\mathsf{f}(Q_n) = 2^{n-1} + A(n,4)$ if and only if Q_n has a maximum induced forest F such that $Q_n - F$ is an empty graph.

Another class of graphs for which the decycling number has been studied to some precision are the grid graphs $P_m \times P_n$, where P_k is the path with k vertices.

The following results were obtained in [5].

Theorem 2.1. *If $m, n \geq 3$, then*

$$\mathsf{f}(P_m \times P_n) \leq \left\lceil \frac{2mn + m + n - 2}{3} \right\rceil.$$

Theorem 2.2. *For $n \geq 4$,*

1. $\mathsf{f}(P_2 \times P_n) = \lceil \frac{3n}{2} \rceil$;
2. $\mathsf{f}(P_3 \times P_n) = \lceil \frac{9n}{4} \rceil$;
3. $\mathsf{f}(P_4 \times P_n) = 3n$;
4. $\mathsf{f}(P_5 \times P_n) = \lceil \frac{7n}{2} \rceil + \lfloor \frac{n}{8} \rfloor + 1$;
5. $\mathsf{f}(P_6 \times P_n) = \lceil \frac{13n}{3} \rceil$;
6. $\mathsf{f}(P_7 \times P_n) = 5n + 1$.

Theorem 2.3. *Let $m = 6q + r$ and $n = 6s + t$ with $1 \leq r, t \leq 6$. Then*

$$\mathsf{f}(P_m \times P_n) \geq \max\{\mathsf{f}(P_r \times P_n) - q(2n-1), \mathsf{f}(P_t \times P_m) - s(2m-1)\}.$$

Theorem 2.4. *For $m, n \geq 2$,*

$$\mathsf{f}(P_m \times P_n) \geq \frac{2mn}{3} - \frac{8n - m - 4}{3}.$$

Theorem 2.5. *For any positive integers m and n, suppose that $n \equiv 0 (\mathrm{mod}\ 2)$ and $m = 3r + 1$. Then*

$$\mathsf{f}(P_m \times P_n) = (m - r)n + r - 1.$$

Theorem 2.6. *For any positive integers r and s*

$$\mathsf{f}(P_{6r+1} \times P_{4s-1}) = 16rs - 2r - 4s - 2.$$

The problem of determining the remaining cases of the grid graphs is open. The following problem is also open.

Problem $f(C_m \times C_m) = ?$

Albertson and Berman [2] conjectured that every planar graph has an induced acyclic subgraph with at least half of the vertices.

Conjecture 2.7. (Albertson and Berman [2]) *Every planar graph has an induced subgraph with at least half of the vertices that is a forest.*

Akiyama and Watanabe [1] gave a similar conjecture in the class of bipartite planar graphs.

Conjecture 2.8. (Akiyama and Watanabe [1]) *Every bipartite planar graph has an induced subgraph with at least $\frac{5}{8}$ of the vertices that is a forest.*

Note that Conjecture 2.7 would directly imply that every planar graph has an independent set with at least one-quarter of the vertices, without using the Four Color Theorem. These questions generalize the independence number in the same way that generalized coloring problems generalize the chromatic number. On the other hand Akiyama and Watanabe [1] gave examples showing that Conjectures 2.7 and 2.8 are best possible.

This is related to a result of Borodin [7] on the *acyclic chromatic number* of a graph, defined to be the minimum number of colors in a proper coloring of the graph in which every 2-chromatic subgraph is acyclic. We denote by $A(G)$ for the acyclic chromatic number of a graph G. Borodin [7] proved the following theorem.

Theorem 2.9. *If G is a planar graph, then $A(G) \leq 5$.*

As a consequence we have the following theorem.

Theorem 2.10. *If G is a planar graph of order n, then $\mathsf{f}(G) \geq \frac{2n}{5}$.*

For outer-planar graphs, Hosono [13] proved the following result and showed in the same paper that this result is best possible.

Theorem 2.11. *If G is an outer-planar graph of order n, then $\mathsf{f}(G) \geq \frac{2n}{3}$.*

3 Connected subclasses

A *switching* in a graph is the replacement of two independent edges by two other independent edges on the same vertices. More precisely, let G be a graph and

$ab, cd \in E(G)$ be independent where $ac, bd \notin E(G)$. A switching $\sigma(a, b; c, d)$ on G is defined by

$$G^{\sigma(a,b;c,d)} = (G - \{ab, cd\}) + \{ac, bd\}.$$

Let G be a graph and $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then $(\deg v_1, \deg v_2, \ldots, \deg v_n)$ is called a *degree sequence* of G. A sequence $\mathbf{d} : (d_1, d_2, \ldots, d_n)$ of nonnegative integers is called a *graphical sequence* if it is a degree sequence of some graph H and in this case H is called a *realization* of $\mathbf{d}$. We write r^n for the sequence $(r, r, \ldots, r)$ of length n. It is well known that r^n is graphical if and only if r is a nonnegative integer, n is a positive integer, $n \geq r + 1$ and $nr \equiv 0 \pmod 2$. A realization of r^n is called an r-regular graph of order n. Furthermore, there exists a disconnected r-regular graph of order n if and only if $n \geq 2r + 2$. We denote $\mathcal{R}(\mathbf{d})$ for the class of all non-isomorphic realizations of $\mathbf{d}$ and $\mathcal{CR}(\mathbf{d})$ for the class of all connected realizations of $\mathcal{R}(\mathbf{d})$. It is clear that a graph obtained from G by a switching on G has the same degree sequence as G.

Let $\mathbf{d}$ be a graphical sequence and $\emptyset \neq \mathcal{J} \subseteq \mathcal{R}(\mathbf{d})$. Then $\mathcal{J}$ is a *connected subclass* of $\mathcal{R}(\mathbf{d})$ if any two distinct realizations $G, H \in \mathcal{J}$, G can be obtained from H by a sequence of switchings so that all intermediate graphs are in $\mathcal{J}$. It was found by Havel [12] and later rediscovered by Hakimi [11] that $\mathcal{R}(\mathbf{d})$ is itself a connected subclass of $\mathcal{R}(\mathbf{d})$. It was proved by Taylor [28] that $\mathcal{CR}(\mathbf{d})$ is a connected subclass of $\mathcal{R}(\mathbf{d})$. Let $\mathcal{B}(r^{2n})$ be the class of all r-regular graphs of order $2n$. It was shown in [30](p. 53) that $\mathcal{B}(r^{2n})$ is a connected subclass of $\mathcal{R}(r^{2n})$.

Let $\mathcal{J}$ be a class of graphs and π be a graph parameter. Then π is said to satisfy an *intermediate value theorem over* $\mathcal{J}$ if $G, H \in \mathcal{J}$ with $\pi(G) < \pi(H)$, then for every integer k with $\pi(G) \leq k \leq \pi(H)$ there is a graph $K \in \mathcal{J}$ such that $\pi(K) = k$. If a graph parameter π satisfies an intermediate value theorem over $\mathcal{J}$, then we write $(\pi, \mathcal{J}) \in \mathsf{IVT}$.

Note that if $(\pi, \mathcal{J}) \in \mathsf{IVT}$, then $\pi(\mathcal{J}) = \{\pi(G) : G \in \mathcal{J}\}$ is uniquely determined by $a(\pi) = \min(\pi, \mathcal{J}) = \min\{\pi(G) : G \in \mathcal{J}\}$ and $b(\pi) = \max(\pi, \mathcal{J}) = \max\{\pi(G) : G \in \mathcal{J}\}$. Then we have $\pi(\mathcal{J}) = \{x \in \mathbb{Z} : a(\pi) \leq x \leq b(\pi)\}$. We proved in [22, 25, 26] that if $\mathcal{J}$ is a connected subclass of $\mathcal{R}(\mathbf{d})$ and $\pi \in \{\mathsf{f}, \chi, \omega\}$, then $(\pi, \mathcal{J}) \in \mathsf{IVT}$.

4 Cubic graphs

The interest of studying the decycling number of cubic graphs was motivated by the results of Zheng and Lu [31], Alon et al. [3], Liu and Zhao [15], and Bau and Beineke [4].

The problems of finding upper bounds of $\mathsf{f}(G)$, where G runs over a class of cubic graphs, have been investigated in the literature. First observe that if G is a cubic graph of order n and F is a maximum induced forest of G, then it is easy to see that $G-F$ is also a forest. Thus $|F| \geq \frac{n}{2}$. The bound is sharp if and only if n is a multiple of 4. We proved in [20] that if $n = 4q+t,\ t = 0,2$, then $\min(\mathsf{f}, 3^n) = 2q$. Let $\mathrm{Min}(\mathsf{f}, 3^n) = \min\{\mathsf{f}(G) : G \in \mathcal{CR}(3^n\}$. The problems of finding $\mathrm{Min}(\mathsf{f}, 3^n)$ are more difficult. A *cubic tree* is a tree whose vertices have degree 1 or 3. Evidently if T is a cubic tree of order n, then $n = 2k+2$, where k is the number of vertices of degree 3 of T. Let K_4' be the graph obtained from K_4 by subdividing one of its edge. Let $\mathcal{T}$ denote the family of cubic graphs obtained by taking cubic trees and replacing each vertex of degree 3 by a triangle and identifying each vertex of degree 1 to the vertex of degree 2 in a copy of K_4'.

A lower bound for the order of maximum induced forest in connected cubic graphs has been obtained by Liu and Zhao [15] as stated in the following theorem.

Theorem 4.1. *Let G be a connected cubic graph of order $n \geq 12$. Then $\mathsf{f}(G) = \frac{5}{8}n - \frac{1}{4}$ if $G \in \mathcal{T}$ and $\mathsf{f}(G) \geq \frac{5}{8}n$ if $G \notin \mathcal{T}$.*

We have determined in [20] the value of $\mathrm{Min}(\mathsf{f}, 3^n)$ by observing the following situation. First observe that if $G \in \mathcal{T}$, then G has order $8k+10$, where k is the number of vertices of degree 3 in the corresponding cubic tree. Thus $\mathsf{f}(G) = \mathrm{Min}(\mathsf{f}, 3^{8k+10}) = 5k+6$. We now consider a cubic graph of order $8k+8$. Let C be a cubic graph of order $8k+8$. Then by Theorem 4.1, $\mathsf{f}(C) \geq \frac{5}{8}(8k+8) = 5(k+1)$. A cubic graph T obtained by taking cubic tree with k vertices of degree 3, replacing $k-1$ of the vertices by a triangle and attaching a copy of K_4' at every vertex of degree 1. Thus T has order $8k+8$ and $\mathsf{f}(T) = 5(k+1)$. Thus $\mathrm{Min}(\mathsf{f}, 3^{8k+8}) = 5(k+1)$. The value of $\mathrm{Min}(\mathsf{f}, 3^n)$, $n = 8k+4, 8k+6$ can be obtained in the following argument. Since a switching changes the order of induced forest by at most 1, we have $\mathrm{Min}(\mathsf{f}, 3^{p+q}) \leq \mathrm{Min}(\mathsf{f}, 3^p) + \mathrm{Min}(\mathsf{f}, 3^q) + 1$ for all even integers p and q with $4 \leq p \leq q$. Thus $5k+4 = \lceil \frac{5}{8}(8k+6) \rceil \leq \mathrm{Min}(\mathsf{f}, 3^{8k+6}) \leq \mathrm{Min}(\mathsf{f}, 3^4) + \mathrm{Min}(\mathsf{f}, 3^{8(k-1)+10}) + 1 = 2 + 5(k-1) + 6 + 1 = 5k+4$.

Finally $5k+3 = \lceil \frac{5}{8}(8k+4) \rceil \leq \mathrm{Min}(\mathsf{f}, 3^{8k+4}) \leq \mathrm{Min}(\mathsf{f}, 3^4) + \mathrm{Min}(\mathsf{f}, 3^{8(k-1)+8}) + 1 = 2 + 5k + 1 = 5k + 3$. Therefore we obtained in [20] the following theorems.

Theorem 4.2. *Let n be an even integer with $n \geq 12$. Then*

$$\mathrm{Min}(\mathsf{f}, 3^n) = \begin{cases} \frac{5}{8}n - \frac{1}{4} & \textit{if } n \equiv 2 (\mathrm{mod}\ 8), \\ \lceil \frac{5}{8}n \rceil & \textit{otherwise}. \end{cases}$$

In [27], there are five connected cubic graphs of order 8, all of which having maximum induced forests of order 5. Alon et al. proved in [3] that if G is a $\{K_4, K_4'\}$-free graph with maximum degree 3, order n and of size m, then $\mathsf{f}(G) \geq n - \frac{m}{4}$. Consequently, if G is a cubic $\{K_4, K_4'\}$-free graph of order $n \geq 10$, then $\mathsf{f}(G) \geq \frac{5n}{8}$. Zheng and Lu proved in [31] that $\mathsf{f}(G) \geq \frac{2n}{3}$ for any connected cubic graph G of order n without triangles, except for two cubic graphs with $n = 8$. They also pointed out that this lower bound is best possible. It is easy to see that there exists a cubic graph G of order n containing triangles and $\mathsf{f}(G) \geq \frac{2n}{3}$. We have extended their result by proving that $\mathsf{f}(G) \geq \frac{2n}{3}$ for any connected cubic K_4'-free graph G of order $n \geq 10$ as stated in the following results.

Theorem 4.3. *Let G be a connected triangle-free graph of order n and $\Delta(G) = 3$. If G is not a cubic graph, then $\mathsf{f}(G) \geq \frac{2n}{3}$.*

Theorem 4.4. *Let $X = \mathcal{CR}(3^8) \cup \{K_4, K_4'\}$ and let G be an X-free graph of order n with $\Delta(G) = 3$. Then $\mathsf{f}(G) \geq \frac{2n}{3}$.*

Theorem 4.5. *Let G be a connected cubic K_4'-free graph of order n, $n \geq 6$ and $n \neq 8$. Then $\mathsf{f}(G) \geq \frac{2n}{3}$.*

We constructed in [8] a class of connected triangle-free graphs $\mathcal{J}_n$ of order n to show that $\min(\mathsf{f}, \mathcal{J}_n) = \lceil \frac{2n}{3} \rceil$ as follows.

First observe that

1. $\mathsf{f}(K_{3,3}) = 4$.
2. $\mathsf{f}(Q_3) = 5$, where Q_3 is the 3-cube.
3. There is a switching σ such that $(2K_4')^\sigma$ is a connected triangle-free graph and $\mathsf{f}((2K_4')^\sigma) = 7$. Put $K = (2K_4')^\sigma$.

4. If $e \in E(K_{3,3})$ and $f \in E(Q_3)$, then $\mathsf{f}(K_{3,3}-e)=4$ and $\mathsf{f}(Q_3-f)=6$. Put $P=K_{3,3}-e$ and $Q=Q_3-f$.

5. Let n be an even integer with $n \geq 12$. Write $n=6q+t$, $t=0,2,4$ and construct a connected cubic triangle-free graph according to the values of t

   5.1 If $t=0$, construct graph G of order $6q$ by taking q copies of P and joining q appropriate edges between the q copies of P.

   5.2 If $t=2$, construct graph G of order $6q+2$ by taking $q-1$ copies of P and a copy of Q and then joining q appropriate edges between them.

   5.3 If $t=4$, construct a graph G of order $6q+4$ by taking $q-1$ copies of P and a copy of K and then joining q appropriate edges between them.

6. It is easy to check that the graphs G constructed above satisfying $\mathsf{f}(G)=\lceil\frac{2n}{3}\rceil$.

Thus we have the following theorem.

Theorem 4.6. [8] $\min(\mathsf{f},\mathcal{J}_n)=\lceil\frac{2n}{3}\rceil$.

In 2002, the following problems were posed in [4]. We rewrite the problems in terms of the forest number.

Problem 1. Which cubic graphs G of order $2n$ satisfy $\mathsf{f}(G)=\lfloor\frac{3n-1}{2}\rfloor$?

Problem 2. Which cubic planar graphs G of order $2n$ satisfy $\mathsf{f}(G)=\lfloor\frac{3n-1}{2}\rfloor$?

We solved the problems in [20] and [21], respectively, as stated in the following results.

Theorem 4.7. [20] *Let G be a cubic graph of order $2n$, then $\mathsf{f}(G)\leq\lfloor\frac{3n-1}{2}\rfloor$.*

Theorem 4.8. [21] *Let G be a cubic planar graph of order $2n$, then $\mathsf{f}(G)\leq\lfloor\frac{3n-1}{2}\rfloor$.*

We have also construct graphs to show that the bound in above theorems are sharp. Let $\mathcal{R}(3^{2n};\ \lfloor\frac{3n-1}{2}\rfloor)$ be the class of cubic graphs G of order $2n$ with $\mathsf{f}(G)=\lfloor\frac{3n-1}{2}\rfloor$. Some characterization of graphs in $\mathcal{R}(3^{2n};\ \lfloor\frac{3n-1}{2}\rfloor)$ were obtained in [20]. In addition we proved the following results.

Lemma 4.9. *Let G be a cubic graph of order $2n$, n is an odd integer, with $\mathsf{f}(G) = \frac{3n-1}{2}$. If G has a path P_N as a maximum induced forest, where $N = \frac{3n-1}{2}$, then G_N can be obtained from G by a finite sequence of switchings $\sigma_1, \sigma_2, \ldots, \sigma_k$ such that for all $i = 1, 2, \ldots, k$, $G^{\sigma_1\sigma_2\ldots\sigma_i}$ is a cubic graph with P_N as its induced forest.*

Lemma 4.10. *Let G be a cubic graph of order $2n$, n is an odd integer, with $\mathsf{f}(G) = \frac{3n-1}{2}$. If G does not have P_N as its maximum induced forest. Then a cubic graph G_N can be obtained from G by a finite sequence of switchings $\sigma_1, \sigma_2, \ldots, \sigma_k$ such that for all $i = 1, 2, \ldots, k$, $G^{\sigma_1\sigma_2\ldots\sigma_i} \in \mathcal{R}(3^{2n};\ \frac{3n-1}{2})$ and $G^{\sigma_1\sigma_2\ldots\sigma_k} = G_N$.*

Similar argument can be made to obtain the same result for cubic graphs of order $2n$ and n is even.

Combining the results in this section, we have the following theorem.

Theorem 4.11. $(\mathsf{f}, \mathcal{R}(3^{2n},\ \lfloor\frac{3n-1}{2}\rfloor)) \in \mathsf{IVT}$.

We continued in investigating in [21] all cubic planar graphs G of order $2n$ with $\mathsf{f}(G) = \lfloor\frac{3n-1}{2}\rfloor$. Let $\mathcal{P}(3^{2n})$ be the class of all connected cubic planar graphs of order $2n$ and $\mathcal{P}(3^{2n},\ \lfloor\frac{3n-1}{2}\rfloor)$ be the class of cubic planar graphs G of order $2n$ with $\mathsf{f}(G) = \lfloor\frac{3n-1}{2}\rfloor$. We can ask the following questions.

1. Is $(\mathsf{f}, \mathcal{P}(3^{2n})) \in \mathsf{IVT}$?

2. What are $\min\{\mathsf{f}(G) : G \in \mathcal{P}(3^{2n})\}$ and $\max\{\mathsf{f}(G) : G \in \mathcal{P}(3^{2n})\}$?

3. Which cubic planar graphs G of order $2n$ satisfy $\mathsf{f}(G) = \lfloor\frac{3n-1}{2}\rfloor$? (Problem 2)

4. Is $\mathcal{P}(3^{2n},\ \lfloor\frac{3n-1}{2}\rfloor)$ a connected subclass?

We answered all of the questions in [21]. Most of the proofs are constructed and we state the results as follows.

Theorem 4.12. (Question 1) $(\mathsf{f}, \mathcal{P}(3^{2n})) \in \mathsf{IVT}$.

Theorem 4.13. (Question 2)

$$\min\{\mathsf{f}(G) : G \in \mathcal{P}(3^{2n})\} = \mathrm{Min}(\mathsf{f}, 3^{2n}) \text{ and}$$
$$\max\{\mathsf{f}(G) : G \in \mathcal{P}(3^{2n})\} = \max(\mathsf{f}, 3^{2n}).$$

We constructed all cubic planar graphs G of order $2n$ with $\mathsf{f}(G) = \frac{3n-1}{2}$ in [21] and proved the following result as an answer to the question 2.

Theorem 4.14. $\mathcal{P}(3^{2n}, \lfloor \frac{3n-1}{2} \rfloor)$ *is a connected subclass.*

Several subclasses of $\mathcal{CR}(3^n)$ have been further investigated in [8]. In particular Let $\mathcal{J}_1 = \mathcal{CR}(3^n)$ and $\mathcal{J}_2$ be the class of connected cubic K_4'-free graphs of order n, where K_4' is a graph obtained from K_4 and a subdivision to an edge. Put $\mathcal{J}_3$ for the class of connected cubic triangle-free graphs of order n. It is clear that $\mathcal{J}_3 \subseteq \mathcal{J}_2 \subseteq \mathcal{J}_1$. Let $X_n = \{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_1 - \mathcal{J}_2, \mathcal{J}_1 - \mathcal{J}_3, \mathcal{J}_2 - \mathcal{J}_3\}$.
We proved in [8] the following results.

Theorem 4.15. *If* $\mathcal{J} \in X_n$*, then* $(\mathsf{f}, \mathcal{J}) \in \mathsf{IVT}$.

Theorem 4.16. *Let* $X_n = \{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_1 - \mathcal{J}_2, \mathcal{J}_1 - \mathcal{J}_3, \mathcal{J}_2 - \mathcal{J}_3\}$. *Then*

1. $\max(\mathsf{f}, \mathcal{J}_1) = \max(\mathsf{f}, \mathcal{J}_2) = \max(\mathsf{f}, \mathcal{J}_3) = \lfloor \frac{3n-2}{4} \rfloor$,

2. $\max(\mathsf{f}, \mathcal{J}_1 - \mathcal{J}_3) = \max(\mathsf{f}, \mathcal{J}_2 - \mathcal{J}_3) = \lfloor \frac{3n-2}{4} \rfloor$,

3. $\max(\mathsf{f}, \mathcal{J}_1 - \mathcal{J}_2) = \lfloor \frac{3n-4}{4} \rfloor$,

4. *Let* n *be an even integer with* $n \geq 12$. *If* $\mathcal{J} \in \{\mathcal{J}_1, \mathcal{J}_1 - \mathcal{J}_2, \mathcal{J}_1 - \mathcal{J}_2\}$*, then*

$$\min(\mathsf{f}, \mathcal{J}) = \begin{cases} \frac{5}{8}n - \frac{1}{4} & if\ n \equiv 2 (\mathrm{mod}\ 8), \\ \lceil \frac{5}{8}n \rceil & otherwise, \end{cases}$$

5. *Let* G *be a connected cubic* K_4'*-free graph of order* $n \neq 8$. *Then* $\mathsf{f}(G) \geq \frac{2n}{3}$,

6. $\min(\mathsf{f}, \mathcal{J}_2) = \min(\mathsf{f}, \mathcal{J}_3) = \min(\mathsf{f}, \mathcal{J}_2 - \mathcal{J}_3) = \lceil \frac{2n}{3} \rceil$.

5 Regular Graphs

As the results Havel [12], Taylor in [28] and the fact that if G is a graph and σ is a switching on G, then $|\mathsf{f}(G) - \mathsf{f}(G^\sigma)| \leq 1$, we obtained that $(\mathsf{f}, \mathcal{J}) \in \mathsf{IVT}$ where $\mathcal{J} \in \{\mathcal{R}(\mathbf{d}), \mathcal{CR}(\mathbf{d})\}$. Thus $\mathsf{f}(\mathcal{J})$ is uniquely determined by $\min(\mathsf{f}, \mathcal{J})$ and $\max(\mathsf{f}, \mathcal{J})$.

By using *the probabilistic method*, we found in [24] a lower bound of $\min(\mathsf{f}, \mathbf{d})$. In particular we proved the following theorem.

Theorem 5.1. *Let G be a graph with degree sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \ldots d_n \geq 1$. Then*

$$\mathsf{f}(G) \geq 2\sum_{i=1}^{n} \frac{1}{d_i + 1}.$$

As a consequence of the theorem and elementary arithmetic, we obtained the following results.

Corollary 5.2. *If the average degree of G is at most d, then $\mathsf{f}(G) \geq \frac{2n}{d+1}$.*

Corollary 5.3. *Let G be an r-regular graph of order n. Then $\mathsf{f}(G) \geq \frac{2n}{r+1}$.*

We further proved that the bound is sharp in the class of graphs of order n and of maximum degree $\Delta = d_1$.

Theorem 5.4. *Let $\mathbf{d} = (d_1, d_2, \ldots, d_n), d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ be a graphical sequence and $d_1 + 1 \leq n \leq 2d_1 + 1$. Then*

1. $\min(\mathsf{f}, \mathbf{d}) = 2$ *if and only if* $d_1 = d_2 = d_3 = \cdots = d_n$ *and* $n = d_1 + 1$ *and*

2. *if* $\mathbf{d}$ *does not have a complete graph as its realization, then* $\min(\mathsf{f}, \mathbf{d}) = 3$ *if and only if* $\overline{\mathbf{d}}$ *has a union of stars as its realization.*

Let $\mathcal{G}(\Delta, n)$ be the class of graphs of order n and of maximum degree Δ. We proved in [22] the following results.

Theorem 5.5. *Let $n = (\Delta + 1)q + t, 0 \leq t \leq \Delta$. Then*

1. $\min(\mathsf{f}, \mathcal{G}(\Delta, n)) = 2q$, *if* $t = 0$,

2. $\min(\mathsf{f}, \mathcal{G}(\Delta, n)) = 2q + 1$, *if* $t = 1$, *and*

3. $\min(\mathsf{f}, \mathcal{G}(\Delta, n)) = 2q + 2$, *if* $2 \leq t \leq \Delta$.

Theorem 5.6. *For $r \geq 3$, and $n = r + j$, $1 \leq j \leq r + 1$*

1. $\min(\mathsf{f}, r^n) = 2$ *if and only if* $n = r + 1$,

2. $\min(\mathsf{f}, r^n) = 3$ *if and only if* $n = r + 2$,

3. $\min(\mathsf{f}, r^n) = 4$, *for all even integers* n, $r + 3 \leq n$,

4. $\min(\mathsf{f}, r^n) = 4$, *for all odd integers* n, $r + 3 \leq n$ *and* $n \geq f(j)$,

5. $\min(\mathsf{f}, r^n) = 5$, *for all odd integers* n, $r+3 \leq n$ *and* $n < f(j)$,
   where $f(j) = \frac{5}{2}(j-1)$ *if* $j \equiv 3 (\bmod 4)$, *and*
   $f(j) = 1 + \frac{5}{2}(j-1)$ *if* $j \equiv 1 (\bmod 4)$.

Theorem 5.7. *For* $n \geq 2r+2$ *and* $r \geq 3$, *write* $n = (r+1)q + t, q \geq 2$ *and* $0 \leq t \leq r$. *Then*

1. $\min(\mathsf{f}, r^n) = 2q$ *if* $t = 0$,
2. $\min(\mathsf{f}, r^n) = 2q + 1$ *if* $t = 1$,
3. $\min(\mathsf{f}, r^n) = 2q + 2$ *if* $2 \leq t \leq r-1$,
4. $\min(\mathsf{f}, r^n) = 2q + 3$ *if* $t = r$.

We obtained in [22] the values of $\max(\mathsf{f}, r^n)$, for all r and n as stated in the following theorem.

Theorem 5.8.

$$\max(\mathsf{f}, r^n) = \begin{cases} n - r + 1 & if\ r+1 \leq n \leq 2r-1, \\ \lfloor \frac{nr-2}{2(r-1)} \rfloor & if\ n \geq 2r. \end{cases}$$

Let $\mathrm{Min}(\mathsf{f}, r^n) = \min\{\mathsf{f}(G) : G \in \mathcal{CR}(r^n)\}$ and $\mathrm{Max}(\mathsf{f}, r^n) = \max\{\mathsf{f}(G) : G \in \mathcal{CR}(r^n)\}$. We obtained the values of $\mathrm{Min}(\mathsf{f}, r^n)$ and $\mathrm{Max}(\mathsf{f}, r^n)$ in [23] as stated in the following results.

Theorem 5.9. $\mathrm{Max}(\mathsf{f}, r^n) = \max(\mathsf{f}, r^n)$ *in all situations.*

Theorem 5.10. $\mathrm{Min}(\mathsf{f}, r^n) = \min(\mathsf{f}, r^n)$ *for all* r *and* n *with* $r+1 \leq n \leq 2r+1$.

In order to obtain the value of $\mathrm{Min}(\mathsf{f}, r^n)$ in other cases, we proved in [23] the following theorem.

Theorem 5.11. *Let* G *be a connected* r*-regular graph of order* $n \geq 2r+2$. *Then* $\mathsf{f}(G) \geq \frac{2n}{r}$ *for all* $r \geq 4$.

Note that we improved a lower bound for $\mathrm{Min}(\mathsf{f}, r^n)$ from $\frac{2n}{r+1} = \min(\mathsf{f}, r^n)$, for $r \geq 4$ and n with $n \geq 2r+2$, to $\frac{2n}{r}$.

Theorem 5.12. *Let* r *and* n *be integers with* $r \geq 4$ *and* $n \geq 2r+2$. *Put* $n = rq + t$, $0 \leq t \leq r-1$. *Then*

$$\mathrm{Min}(\mathsf{f}, r^n) = \begin{cases} 2q & if\ t = 0, \\ 2q+1 & if\ t = 1, 2, \\ 2q+2 & if\ \frac{r}{2} \le t \le r-1. \end{cases}$$

Theorem 5.13. *Let r and n be integers with $r \ge 4$ and $n \ge 2r+2$. Put $n = rq + t$, $0 \le t \le r-1$. Then* $\mathrm{Min}(\mathsf{f}, r^n) \in \{2q+1, 2q+2\}$ *if* $3 \le t \le \frac{r}{2}$.

Conjecture 5.14. *Let r and n be integers with $r \ge 4$ and $n \ge 2r+2$. Put $n = rq + t$, $0 \le t \le r-1$. Then* $\mathrm{Min}(\mathsf{f}, r^n) = 2q+2$ *if* $3 \le t \le \frac{r}{2}$.

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