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Almost Quasi-mininjective Modules

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Abstract: Let M be a right R-module, $S = \text{End}_R(M)$. The module M is called *almost quasi-mininjective* (or $AQ-$ *mininjective*) if, for any simple $M-$cyclic submodule $s(M)$ of M, there exists a left ideal X_s of S such that $l_S(Ker(s)) = Ss \bigoplus X_s$ as left $S-$modules. In this paper, we give some characterizations and properties of $AQ-$ mininjective modules.

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1 Introduction

Let R be a ring. A right R-module M is called *principally injective* (or P-*injective*) if, every R-homomorphism from a principal right ideal of R to M can be extended to an R-homomorphism from R to M. Equivalently, $l_M(r_R(a)) = Ma$ for all $a \in R$, where l_M and r_R are the left and right annihilators in M and R, respectively. In [5], Nicholson and Yousif studied the structure of principally injective rings and gave some applications. They also continued to study rings with some other kind of injectivity, namely, mininjective rings [6]. A right R-module M is called *mininjective* if, every $R-$ homomorphism from a simple right ideal of R to M can be extended to an $R-$ homomorphism from R to M, or equivalently, if kR is simple, $k \in R$, $l_M(r_R(k)) = Mk$. If the regular right R-module R_R is mininjective, then the ring R is said to be a *right mininjective ring.* In [10], right mininjective rings are generalized to almost mininjective rings, that is, a right R-module M is called *almost mininjective* (or *$A-$ mininjective*) if, for any simple right ideal kR of R, there exists an S-submodule X_k of M such that $l_M(r_R(k)) = Mk \bigoplus X_k$ as left S-modules. If R_R is an almost mininjective module, then we call R is a *right almost mininjective ring.* The nice structure of almost mininjective rings draws our attention to define AQ-mininjective modules, and to investigate their characterizations and properties.

Throughout this paper, R will be an associative ring with identity and all modules are unitary right R-modules. For right R-modules M and N, $\text{Hom}_R(M,N)$ denotes the set of all R-homomorphisms from M to N and $S = \text{End}_R(M)$. A submodule N of M is said to be an *$M-$ cyclic submodule* of M if it is the image of an element of S. By notation $N \subset^{\oplus} M$ ($N \subset^{e} M$) we mean that N is a direct summand (an essential submodule) of M. We denote the socle and the singular submodule of M by $Soc(M)$ and $Z(M)$, respectively, and that $J(M)$ denotes the Jacobson radical of M.

Following [7], for an R-module N and a submodule P of N, we will identify $\text{Hom}_R(N,M)$ with the set of maps in $\text{Hom}_R(P,M)$ that can be extended to N, and hence $\text{Hom}_R(N,M)$ becomes a left S-submodule of $\text{Hom}_R(P,M)$. In particular, for an element $s \in S$, S will be regarded as a left S-submodule of $\text{Hom}_R(s(M),M)$.

2 AQ-mininjective Modules

Definition 2.1. Let M be a right R-module, $S = \text{End}_R(M)$. The module M is called *almost quasi-mininjective* (or *$AQ-$ mininjective*) if, for any simple $M-$cyclic submodule $s(M)$ of M, there exists a left ideal X_s of S such that $l_S(Ker(s)) = Ss \bigoplus X_s$ as left $S-$modules.

Lemma 2.2. *Let M be a right R-module and let $s(M)$ be an $M-$ cyclic submodule of M.*

(1) If $\text{Hom}_R(s(M),M) = S \bigoplus Y$ as left S-modules, then $l_S(Ker(s)) = Ss \bigoplus X$ as left S-modules, where $X = \{fs : f \in Y\}$.

(2) If $l_S(Ker(s)) = Ss \bigoplus X$ for some $X \subset S$ as left S-modules, then we have $\text{Hom}_R(s(M),M) = S \bigoplus Y$ as left S-modules, where $Y = \{f \in \text{Hom}_R(s(M),\ M) : fs \in X\}$.

(3) *Ss is a direct summand of $l_S(Ker(s))$ as left S-modules if and only if S is a direct summand of* $\mathrm{Hom}_R(s(M), M)$ *as left S-modules.*

Proof. Define $\theta : \mathrm{Hom}_R(s(M), M) \to l_S(Ker(s))$ by $\theta(f) = fs$ for every $f \in \mathrm{Hom}_R(s(M), M)$. It is obvious that θ is an S-monomorphism. For $t \in \ell_S(Ker(s))$, define $g : s(M) \to M$ by $g(s(m)) = t(m)$ for every $m \in M$. Since $Ker(s) \subset Ker(t)$, g is well-defined, so it is clear that g is an R-homomorphism. Then $\theta(g) = gs = t$. Therefore θ is an S-isomorphism. Let $fs \in Ss$. Since $fs \in l_S(Ker(s))$, there exists $\varphi \in Hom_R(s(M), M)$ such that $\theta(\varphi) = fs$, so $\varphi s = fs$. Define $\widehat{\varphi} : M \to M$ by $\widehat{\varphi}(m) = f(m)$ for every $m \in M$. It is clear that $\widehat{\varphi}$ is an R-homomorphism and is an extension of φ. Then $fs = \widehat{\varphi} s = \theta(\widehat{\varphi})$. This shows that $Ss \subset \theta(S)$. The other inclusion is clear. Then $\theta(S) = Ss$ and $X = \theta(Y) = \{fs : f \in Y\}$. Then the lemma follows. □

From Lemma 2.2, the following corollary follows.

Corollary 2.3. *Let M be a right R-module and let $s(M)$ be an $M-$cyclic submodule of M. Then $l_S(Ker(s)) = Ss$ if and only if every R-homomorphism from $s(M)$ to M can be extended to M.*

Theorem 2.4. *The following conditions are equivalent:*

(1) *M is $AQ-$mininjective.*

(2) *There exists an indexed set $\{X_s : s \in S\}$ of left ideals of S with the property that if $s(M)$ is simple, $s \in S$, then $\ell_S(Im(t) \cap Ker(s)) = (X_{st} : t)_l + Ss$ and $(X_{st} : t)_l \cap Ss \subset l_S(t)$ for all $t \in S$, where $(X_{st} : t)_l = \{g \in S : gt \in X_{st}\}$ if $st \neq 0$ and $X_0 = 0$.*

Proof. (1)⇒(2) Let $s(M)$ be a simple M-cyclic submodule of M. Then there exists a left ideal X_s of S such that $l_S(Ker(s)) = Ss \bigoplus X_s$ as left $S-$modules. Let $t \in S$. If $st \neq 0$, then for any $g \in \ell_S(Im(t) \cap Ker(s))$ we have $Ker(st) \subset Ker(gt)$. Since $s(M)$ is simple, $st(M) = s(M)$. Then there exists a left ideal X_{st} of S such that $l_S(Ker(st)) = Sst \bigoplus X_{st}$ as left $S-$modules. Thus $gt \in Sst \bigoplus X_{st}$ because $gt \in l_S(Ker(gt)) \subset l_S(Ker(st))$. Write $gt = fst + x$ where $f \in S$ and $x \in X_{st}$. Then $(g - fs)t = x \in X_{st}$, so $g - fs \in (X_{st} : t)_l$. It follows that $g \in (X_{st} : t)_l + Ss$. This shows that $\ell_S(Im(t) \cap Ker(s)) \subset (X_{st} : t)_l + Ss$. Conversely, it is clear that $Ss \subset \ell_S(Im(t) \cap Ker(s))$. Let $y \in (X_{st} : t)_l$. Then $yt \in X_{st} \subset l_S(Ker(st))$. If $t(m) \in Im(t) \cap Ker(s)$, then $st(m) = 0$ and so $yt(m) = 0$.

Hence $y \in \ell_S(Im(t) \cap Ker(s))$. This shows that $(X_{st} : t)_l \subset \ell_S(Im(t) \cap Ker(s))$. Therefore $\ell_S(Im(t) \cap Ker(s)) = (X_{st} : t)_l + Ss$. If $gs \in (X_{st} : t)_l \cap Ss$, then $gst \in X_{st} \cap Sst = 0$. Hence $gs \in l_S(t)$.

(2)$\Rightarrow$(1) Let $s(M)$ be a simple $M-$cyclic submodule of M. Then there exists a left ideal X_s of S such that $l_S(ker(s)) = \ell_S(Im(1) \cap Ker(s)) = (X_s : 1)_l + Ss$ and $(X_s : 1)_l \cap Ss \subset l_S(1) = 0$. Note that $(X_s : 1)_l = X_s$. Then (1) follows. □

Note that, the ring R is right almost mininjective if and only if R_R is AQ-mininjective. From this result and Theorem 2.4 we have

Corollary 2.5. [10, Theorem 3.1] *The following conditions are equivalent:*

(i) *R is right A-mininjective.*

(ii) *There exists an indexed set $\{X_a : a \in R\}$ of left ideals of R with the property that if kR is simple, $k \in R$, then $\ell[aR \cap r(k)] = (X_{ka} : a)_l + Rk$ and $(X_{ka} : a)_l \cap Rk \subset l(a)$ for all $a \in R$, where $(X_{ka} : a)_l = \{x \in R : xa \in X_{ka}\}$ if $ka \neq 0$ and $X_0 = 0$.*

Following [6], we consider the conditions MC_2 and MC_3 for a ring R.

MC_2: If $kR \simeq eR$ is simple, $e = e^2$, then $kR = gR$, for some $g = g^2$.

MC_3: If $eR \neq fR$ are simple, $e = e^2, f = f^2$, then $eR \bigoplus fR = gR$ for some $g = g^2$.

The next proposition shows that the conditions (MC_2) and (MC_3) also hold in an $AQ-$mininjective module.

Proposition 2.6. *Let M be an $AQ-$mininjective module and $S = \mathrm{End}_R(M)$.*

(1) *If $e(M) \simeq k(M)$ is simple, $e^2 = e \in S$, then $k(M) = g(M)$, for some $g^2 = g \in S$.*

(2) *If $e(M) \neq f(M)$ are simple, $e^2 = e \in S, f^2 = f \in S$, then $e(M) \bigoplus f(M) = g(M)$ for some $g^2 = g \in S$.*

Proof. (1) Let $e(M) \simeq k(M)$ is a simple submodule of M, $e^2 = e \in S$ and let $\sigma : e(M) \to k(M)$ be an $R-$isomorphism. Set $\alpha = \sigma e$. Then $\alpha(M) = k(M)$ and $Ker(e) = Ker(\alpha)$, so $\alpha(M)$ is a simple submodule of M. Then $e \in l_S(Ker(e)) = l_S(Ker(\alpha)) = S\alpha \bigoplus X_\alpha$ where X_α is a left ideal of S. Write $e = s\alpha + x$ where $s \in S$ and $x \in X_\alpha$. Thus $\alpha = \alpha e = \alpha s\alpha + \alpha x$ and so $\alpha - \alpha s\alpha = \alpha x \in S\alpha \cap X_\alpha = 0$, hence $\alpha = \alpha s\alpha$. Put $g = \alpha s$. Then $g^2 = g$ and $k(M) = g(M)$.

(2) Let $e(M) \neq f(M)$ are simple, $e^2 = e \in S, f^2 = f \in S$. Then we have $e(M) \bigoplus f(M) = e(M) \bigoplus (1-e)f(M)$. If $(1-e)f(M) = 0$, then $e(M) \bigoplus f(M) = e(M)$, because by assumption we have $e(M) \cap f(M) = 0$. Hence $e(M) \bigoplus f(M)$ is a direct summand of M. If $(1-e)f(M) \neq 0$, then $f(M) \simeq (1-e)f(M)$ so $(1-e)f(M) = g(M)$, $g^2 = g \in S$, by (1). Then $eg = 0$ so $h = e + g - ge$ is an idempotent such that $he = e = eh$ and $hg = g = gh$. If $x \in e(M) \bigoplus f(M)$, then $x \in e(M) \bigoplus (1-e)f(M) = e(M) \bigoplus g(M)$. Write $x = e(m) + g(n)$. It follows that $x = he(m) + hg(n) = h(e(m) + g(n)) \in h(M)$. This shows that $e(M) \bigoplus f(M) \subset h(M)$. The other inclusion is clear. Then $e(M) \bigoplus f(M) = h(M)$. □

Proposition 2.7. *Let M be an $AQ-$mininjective module which is a principal self-generator. Then $Soc(M_R) \subset r_M(J(S))$.*

Proof. Let mR be a simple submodule of M. Suppose $\alpha(m) \neq 0$ for some $\alpha \in J(S)$. As M is a principal self-generator, $mR = \sum_{s \in I} s(M)$ for some $I \subset S$. Since mR is a simple, $mR = s(M)$ for some $0 \neq s \in I$. Then $\alpha s \neq 0$ and $Ker(\alpha s) = Ker(s)$. Note that $\alpha s(M)$ is a nonzero homomorphic image of the simple module $s(M)$, then $\alpha s(M)$ is simple. Since M is $AQ-$mininjective, there exists a left ideal $X_{\alpha s}$ of S such that $l_S(ker(\alpha s)) = S\alpha s \bigoplus X_{\alpha s}$ as left $S-$modules. Thus $l_S(ker(s)) = S\alpha s \bigoplus X_{\alpha s}$. Write $s = \beta \alpha s + x$ where $\beta \in S$ and $x \in X_{\alpha s}$. Then $(1-\beta\alpha)s = x$ and so $s = (1-\beta\alpha)^{-1}x \in X_{\alpha s}$. It follows that $\alpha s \in S\alpha s \cap X_{\alpha s} = 0$, a contradiction. □

The following corollary follows from Proposition 2.7 and [8, 21.15].

Corollary 2.8. *Let M be an $AQ-$mininjective module which is a principal self-generator. If S is semilocal, then $Soc(M_R) \subset Soc(_S M)$.*

Let M be a right $R-$module with $S = \text{End}_R(M)$. Following [4], write $\triangle = \{s \in S : ker(s) \subset^e M\}$. It is known that $\triangle$ is an ideal of S [4, Lemma 3.2].

Proposition 2.9. *Let M be an $AQ-$mininjective module which is a principal self-generator and $Soc(M_R) \subset^e M$. Then $J(S) \subset \triangle$.*

Proof. Let $s \in J(S)$. If $Ker(s) \not\subset^e M$, then $Ker(s) \cap N = 0$ for some nonzero submodule N of M. Since $Soc(M_R) \subset^e M$, $Soc(M_R) \cap N \neq 0$. Then there exists a simple submodule kR of M such that $kR \subset Soc(M_R) \cap N$ [1, Corollary 9.10]. As M is a principal self-generator and kR is simple, $kR = t(M)$ for some $t \in S$. It follows that $Ker(st) = Ker(t)$. Since $ts(M)$ is a nonzero homomorphic image of the simple module $t(M)$, $st(M) = t(M)$. Then there exists a left ideal X_{st} of S such that $t \in l_S(ker(t)) = l_S(ker(st) = Sst \bigoplus X_{st}$. Write $t = \alpha st + x$ where $\alpha \in S$ and $x \in X_{st}$. It follows that $t = (1 - \alpha s)^{-1}x$. Then $st = s(1 - \alpha s)^{-1}x \in Sst \cap X_{st} = 0$, a contradiction. □

Proposition 2.10. *Let M be an $AQ-$mininjective module which is a principal self-generator and $Soc(M_R) \subset^e M$. If M is nonsingular, then $J(S) = 0$.*

Proof. Since $J(S) \subset \triangle$ by Proposition 2.9, we show that $\triangle = 0$. Let $s \in \triangle$ and let $m \in M$. Define $\varphi : R \to M$ by $\varphi(r) = mr$. It is clear that φ is an $R-$homomorphism. Thus

$$\begin{aligned} r_R(s(m)) &= \{r \in R : s(mr) = 0\} \\ &= \{r \in R : mr \in Ker(s)\} \\ &= \{r \in R : \varphi(r) \in Ker(s)\} \\ &= \varphi^{-1}(Ker(s)). \end{aligned}$$

It follows that $\varphi^{-1}(Ker(s)) \subset^e R$ [3, Lemma 5.8(a)] so $r_R(s(m)) \subset^e R$. Thus $s(m) \in Z(M_R) = 0$ because M is nonsingular. As this is true for all $m \in M$, we have $s = 0$. Hence $\triangle = 0$ as required. □

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