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n-Weak Amenability of T-Lau Product of Banach Algebras

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Abstract: Given a morphism T from a Banach algebra B into a commutative Banach algebra A, we explain explicitly the derivations from $T-$Lau product $A \times_T B$ into its n^{th}-dual $(A \times_T B)^{(n)}$ from which we obtain general necessary and sufficient conditions for $A \times_T B$ to be $n-$weakly amenable.

Keywords: $n-$weak amenability, Lau product, derivation, module action

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1 Introduction

Let A and B be Banach algebras and throughout the paper let A be commutative. Suppose that $T : B \to A$ is an algebra morphism with $\|T\| \leq 1$. Then the direct product $A \times B$ equipped with the ℓ^1-norm and the algebra multiplication

$$(a, b) \cdot (c, d) = (ac + T(d)a + T(b)c, bd), \quad (a, c \in A, b, d \in B),$$

is a Banach algebra which is called the $T-$Lau product of A and B and will denote by $A \times_T B$. Some properties of this algebra such as, Arens regularity, amenability and weak amenability are investigated in [1]. This type of product was introduced by Lau [5] for certain class of Banach algebras and then was extended by Sangani Monfared [6] for the general case. In [5, 6] for two Banach algebras A and B and for $\theta \in \triangle(B)$ (the spectrum of B), $A \times B$, equipped with this type of product so-called "$\theta-$Lau product" was introduced as follows and was denoted by $A \times_\theta B, (a,b)\cdot(c,d) = (ac + \theta(b)c + \theta(d)a, bd)$. An illuminating case which is of special interest is the case $B = \mathbb{C}$ with θ as the identity character i that we get the unitization $A^\sharp = A \times_i \mathbb{C}$ of A. If one includes the possibility that $\theta = 0$ then the usual direct product of Banach algebras will obtain. Besides the works of Lau and Sangani Monfared several properties such as, character inner amenability, biprojectivity and biflatness of $A \times_\theta B$ are investigated in [3, 4].

The main aim of this paper is to study the $n-$weak amenability of $A \times_T B$. In this direction we shall prove that: If A is a commutative Banach algebra, B is a Banach algebra and $T : B \to A$ is a morphism with $\|T\| \leq 1$ then $A \times_T B$ is $(2n+1)-$weakly amenable if and only if both A and B are also. We shall also show that $(2n)-$weak amenability of $A \times_T B$ implies $(2n)-$weak amenability of A and B. Also if A and B are $(2n)-$weakly amenable then $A \times_T B$ is $(2n)-$weakly amenable whenever $\bar{A^2} = A, \bar{B^2} = B$.

2 Preliminaries

A derivation from a Banach algebra A into a Banach $A-$module X is a bounded linear mapping $D : A \to X$ such that for $a, b \in A, D(ab) = D(a)\cdot b + a \cdot D(b)$. The set of all derivations from A into X is denoted by $Z^1(A,X)$. For $x \in X$ the derivation $\delta_x : A \to X$ defined by $\delta_x(a) = a\cdot x - x\cdot a (a \in A)$ is called an inner derivation. The set of all inner derivations from A into X is denoted by $N^1(A,X)$. The quotient $\frac{Z^1(A,X)}{N^1(A,X)}$ that will be denoted by $H^1(A,X)$ is called the first cohomology group of A with coefficients in X. Throughout the paper n is assumed to be a non-negative integer. For a Banach algebra A, the n^{th}-dual $A^{(n)}$ of A is a Banach $A-$module with the module operations that are defined inductively by

$$\langle m\cdot a, f\rangle = \langle m, a\cdot f\rangle,\ \langle a\cdot m, f\rangle = \langle m, f\cdot a\rangle,\ (m \in A^{(n)},\ f \in A^{(n-1)},\ a \in A^{(0)} = A).$$

Also A is a Banach $A-$module under its multiplication. It is clear that in the case where A is commutative $m\cdot a = a\cdot m,\ (a \in A,\ m \in A^{(n)})$.

A Banach algebra A is said to be $n-$weakly amenable if $H^1(A, A^{(n)}) = 0$. This notion was initiated and studied in [2]. Obviously, $1-$weak amenability is nothing else than weak amenability.

For brevity of notation we usually identify an element of A with its canonical image in $A^{(2n)}$. We usually apply $\langle\cdot,\cdot\rangle$ for the duality between a Banach space and its dual and we also use the symbol " $\cdot$ " for the various module operations linking various Banach algebras.

Let $T: B \to A$ be an algebra morphism with $\|T\| \le 1$, and let $T^{(n)}$ be n^{th}-adjoint of T, where $T^{(0)} = T$. It is clear that $\|T^{(n)}\| \le 1$ and also $T^{(2n)} : B^{(2n)} \to A^{(2n)}$ is an algebra morphism extending T to $B^{(2n)}$. To obtain the relation between $n-$weak amenability of $A \times_T B$ and those of A and B we need to characterize the derivations from $A \times_T B$ into $(A \times_T B)^{(n)}$.

3 main results

One can simply identify the underlying space of $(A \times_T B)^{(n)}$ with $A^{(n)} \times B^{(n)}$, equipped with l_1-norm when n is even and the $l_\infty-$norm when n is odd. So one can simply verify that, for $a \in A, b \in B,\ f \in A^{(2n+1)},\ g \in B^{(2n+1)},\ F \in A^{(2n)}$ and $G \in B^{(2n)}$:

$$(a,b)\cdot(f,g) = ((a+T(b))\cdot f,\ T^{(2n+1)}(a\cdot f) + b\cdot g),$$
$$(f,g)\cdot(a,b) = ((a+T(b))\cdot f,\ T^{(2n+1)}(a\cdot f) + g\cdot b),$$
$$(a,b)\cdot(F,G) = ((a+T(b))\cdot F + T^{(2n)}(G)\cdot a,\ b\cdot G),$$
$$(F,G)\cdot(a,b) = ((a+T(b))\cdot F + T^{(2n)}(G)\cdot a,\ G\cdot b).$$

We characterize the derivations from $A \times_T B$ into $(A \times_T B)^{(n)}$.

Note that the fact $m\cdot a = a\cdot m$ $(a \in A, m \in A^{(n)})$ are used repeatedly. The following result is devoted to the case that n is odd.

Proposition 3.1. *Let A be a commutative Banach algebra, B be a Banach algebra, and let $T: B \to A$ be an algebra morphism with $\|T\| \le 1$. A bounded linear mapping $D : A \times_T B \to (A \times_T B)^{(2n+1)}$ is a derivation if and only if there exist derivations $d_A : A \to A^{(2n+1)},\ d_B : B \to B^{(2n+1)}$ and bounded linear mappings $S : A \to B^{(2n+1)}, R : B \to A^{(2n+1)}$ satisfying,*
(i) $D((a,b)) = (d_A(a) + R(b),\ S(a) + d_B(b))$,
(ii) $S(ac) = T^{(2n+1)}(d_A(ac))$,
(iii) $R(bd) = T(b)\cdot R(d) + T(d)\cdot R(b)$,
(iv) $a\cdot R(b) = d_A(T(b))\cdot a$,
(v) $S(a)\cdot b = b\cdot S(a) = T^{(2n+1)}(T(b)\cdot d_A(a))$ $(a, c \in A,\ b, d \in B)$.
In particular, D is inner if and only if $d_A = 0,\ R = 0,\ S = 0$ and d_B is inner.

Proof. For a bounded linear mapping $D : A \times_T B \to (A \times_T B)^{(2n+1)}$, there exist bounded linear mappings $D_1 : A \times_T B \to A^{(2n+1)},\ D_2 : A \times_T B \to B^{(2n+1)}$ such

that $D((a,b)) = (D_1((a,b)),\ D_2((a,b)))$. Define
$d_A(a) = D_1((a,0)), d_B(b) = D_2((0,b)), S(a) = D_2((a,0))$ and $R(b) = D_1((0,b))$.
Let D be a derivation. So for $(a,b),(c,d) \in A \times_T B$ the equality,

$$D((a,b)(c,d)) = (a,b) \cdot D((c,d)) + D((a,b)) \cdot (c,d), \tag{1}$$

is hold. The assumption $b = d = 0$ in (1) implies that d_A is a derivation and $S(ac) = T^{(2n+1)}(d_A(ac)),\ (a,c \in A)$. Also the assumption $a = c = 0$ implies that d_B is a derivation and $R(bd) = T(b) \cdot R(d) + T(d) \cdot R(b),\ (b,d \in B)$.
On the one hand since

$$D((T(b)a,0)) = D((a,0)(0,b)) = D((a,0)) \cdot (0,b) + (a,0) \cdot D((0,b))$$

we have,

$$a \cdot R(b) + T(b) \cdot d_A(a) = d_A(T(b)a) \tag{2}$$

$$S(T(b)a) = S(a) \cdot b + T^{(2n+1)}(R(b) \cdot a),\ (a \in A, b \in B). \tag{3}$$

On the other hand since
$D((T(b)a,0)) = D((0,b)(a,0)) = D((0,b)) \cdot (a,0) + (0,b) \cdot D((a,0))$ we have,

$$S(T(b)a) = T^{(2n+1)}(R(b) \cdot a) + b \cdot S(a),\ (a \in A, b \in B). \tag{4}$$

So by (3) and (4), $S(a) \cdot b = b \cdot S(a)$ and by (2)
$a \cdot R(b) + T(b) \cdot d_A(a) = d_A(T(b)a) = d_A(T(b)) \cdot a + T(b) \cdot d_A(a)$. It follows that $a \cdot R(b) = d_A(T(b)) \cdot a$. Also by (ii) and (4),

$$\begin{aligned} T^{(2n+1)}(R(b) \cdot a) + b \cdot S(a) &= S(T(b)a) \\ &= T^{(2n+1)}(d_A(T(b)a)) \\ &= T^{(2n+1)}(T(b) \cdot d_A(a) + d_A(T(b)) \cdot a) \\ &= T^{(2n+1)}(T(b) \cdot d_A(a)) + T^{(2n+1)}(d_A(T(b)) \cdot a) \\ &= T^{(2n+1)}(T(b) \cdot d_A(a)) + T^{(2n+1)}(R(b) \cdot a). \end{aligned}$$

Hence $S(a) \cdot b = b \cdot S(a) = T^{(2n+1)}(T(b) \cdot d_A(a))$.
A straightforward calculation can be applied to show that the converse is hold.
Let D be an inner derivation. So there exists $(f,g) \in (A \times_T B)^{(2n+1)}$ such that

$D = \delta_{(f,g)}$. It follows that

$$\begin{aligned}(d_A(a) + R(b), S(a) + d_B(b)) &= (a, b) \cdot (f, g) - (f, g) \cdot (a, b) \\ &= ((a + T(b)) \cdot f, T^{(2n+1)}(f \cdot a) + b \cdot g) \\ &\quad - ((a + T(b)) \cdot f, T^{(2n+1)}(f \cdot a) + g \cdot b) \\ &= (0, b \cdot g - g \cdot b) = (0, \delta_g(b)).\end{aligned}$$

It follows that $d_A = 0$, $S = 0$, $R = 0$ and $d_B = \delta_g$. Obviously the converse is hold, indeed if $d_A = 0$, $S = 0$, $R = 0$ and $d_B = \delta_g$ then $D = \delta_{(0,g)}$. □

For the case that n is even we have the next result which needs a similar proof as Proposition 3.1.

Proposition 3.2. *Let A be a commutative Banach algebra, B be a Banach algebra, and let $T : B \to A$ be an algebra morphism with $\|T\| \leq 1$. A bounded linear mapping*
$D : A \times_T B \to (A \times_T B)^{(2n)}$ is a derivation if and only if there exist derivation $d_B : B \to B^{(2n)}$ and bounded linear mappings $d_A : A \to A^{(2n)}, S : A \to B^{(2n)}$ and $R : B \to A^{(2n)}$ satisfying,
(i) $D((a, b)) = (d_A(a) + R(b), S(a) + d_B(b))$,
(ii) $d_A(ac) = a \cdot d_A(c) + d_A(a) \cdot c + T^{(2n)}(s(c)) \cdot a + T^{(2n)}(s(a)) \cdot c$,
(iii) $S(ac) = 0$,
(iv) $d_A(T(b)) \cdot a + T^{(2n)}(S(T(b))) \cdot a = R(b) \cdot a + T^{(2n)}(d_B(b)) \cdot a$,
(v) $R(bd) = T(b) \cdot R(d) + T(d) \cdot R(b)$,
(vi) $S(a) \cdot b = b \cdot S(a) = 0, \quad (a, c \in A,\ b, d \in B)$.
In particular, D is inner if and only if $d_A = 0$, $R = 0$, $S = 0$ and d_B is inner.

Theorem 3.3. *Let A be a commutative Banach algebra, B be a Banach algebra, and let $T : B \to A$ be an algebra morphism with $\|T\| \leq 1$. Then $A \times_T B$ is $(2n+1)-$weakly amenable if and only if both A and B are also.*

Proof. Let $A \times_T B$ be $(2n+1)-$weakly amenable and let $d_A : A \to A^{(2n+1)}$ and $d_B : B \to B^{(2n+1)}$ be derivations. Define $D : A \times_T B \to (A \times_T B)^{(2n+1)}$ by $D = (d_A + R, S + d_B)$ where, $R : B \to A^{(2n+1)}$ and $S : A \to B^{(2n+1)}$ are defined by $R(b) = d_A(T(b))$ and $S(a) = T^{(2n+1)}(d_A(a))$ respectively. Since A is commutative a straightforward calculation shows that the parts $(i), (ii), (iii), (iv)$ in Proposition

3.1 are hold. We show that part (v) is also hold. Let $a \in A,\ b \in B,\ G \in B^{(2n)}$.

$$\begin{aligned}\langle S(a)\cdot b, G\rangle &= \langle S(a), b\cdot G\rangle = \langle T^{(2n+1)}(d_A(a)), b\cdot G\rangle \\ &= \langle d_A(a), T^{(2n)}(b\cdot G)\rangle = \langle d_A(a), T^{(2n)}(b)T^{(2n)}(G)\rangle \\ &= \langle d_A(a), T(b)\cdot T^{(2n)}(G)\rangle = \langle T(b)\cdot d_A(a), T^{(2n)}(G)\rangle \\ &= \langle T^{(2n+1)}(T(b)\cdot d_A(a)), G\rangle.\end{aligned}$$

It follows that $S(a)\cdot b = T^{(2n+1)}(T(b)\cdot d_A(a))$. A similar calculation can be applied to show that $b\cdot S(a) = T^{(2n+1)}(T(b)\cdot d_A(a))$. Hence D is a derivation. Since $A\times_T B$ is $(2n+1)-$weakly amenable, there exists $(f,g)\in (A\times_T B)^{(2n+1)}$ such that $D=\delta_{(f,g)}$.

$$\begin{aligned}D((a,b)) &= (d_A(a)+R(b), S(a)+d_B(b)) = (a,b)\cdot(f,g)-(f,g)\cdot(a,b) \\ &= ((a+T(b))\cdot f, T^{(2n+1)}(a\cdot f)+b\cdot g) \\ &- ((a+T(b))\cdot f, T^{(2n+1)}(a\cdot f)+g\cdot b) \\ &= (0, b\cdot g - g\cdot b).\end{aligned}$$

It follows that $d_A=0$ and $d_B=\delta_g$. So A and B are $(2n+1)-$weakly amenable.
For the converse let A and B be $(2n+1)-$weakly amenable.
So by [[2], Proposition, 1.2], A and B are weakly amenable and it implies that $\bar{A^2}=A$ and $\bar{B^2}=B$.
Let $D: A\times_T B\to (A\times_T B)^{(2n+1)}$ be a derivation. So by Proposition 3.1, there exist derivations $d_A: A\to A^{(2n+1)}$ and $d_B: B\to B^{(2n+1)}$ and also bounded linear mappings
$S: A\to B^{(2n+1)}$ and $R: B\to A^{(2n+1)}$ such that $D=(d_A+R, S+d_B)$. Since A is $(2n+1)-$weakly amenable and commutative, $d_A=0$. Also $d_B=\delta_g$, for some $g\in B^{(2n+1)}$. We show that $S=0$ and $T=0$. Let $a,c\in A$ then by part (ii) of Proposition 3.1,
$S(ac)=T^{(2n+1)}(d_A(ac))=0$. Since S is bounded and $\bar{A^2}=A$, it follows that $S=0$. On the other hand by part (iv) of Proposition 3.1, $a\cdot R(b)=d_A(T(b))\cdot a = 0, a\in A, b\in B$. So the equality $R(bd)=T(b)\cdot R(d)+T(d)\cdot R(b)$ implies that $R(bd)=0$. Since $\bar{B^2}=B$ and R is bounded, $R=0$. So $D=(0,\delta_g)=\delta_{(0,g)}$. □

In the next result we show that under some mild conditions $(2n)-$weak amenability of $A\times_T B$ is equivalent to $2n-$weak amenability of A and B. This condition shows that weak amenability of A and B play a pivotal role for $(2n+1)-$weak amenability $A\times_T B$.

Theorem 3.4. *Let A be a commutative Banach algebra, B be a Banach algebra, and let $T : B \to A$ be an algebra morphism with $\|T\| \leq 1$. Then $(2n)-$weak amenability of $A \times_T B$ implies $(2n)-$weak amenability of both A and B. The converse is hold whenever $\bar{A^2} = A, \bar{B^2} = B$.*

Proof. Let $A \times_T B$ be $(2n)-$weakly amenable and let $d_A : A \to A^{(2n)}$ and $d_B : B \to B^{(2n)}$ be derivations. Define $D : A \times_T B \to (A \times_T B)^{(2n)}$
by $D = (d_A + R, d_B)$, where $R(b) = d_A(T(b)) - T^{(2n)}(d_B(b)),\ b \in B$. Then for each $b, d \in B$,

$$\begin{aligned}
R(bd) &= d_A(T(bd)) - T^{(2n)}(d_B(bd)) = d_A(T(b)T(d)) - T^{(2n)}(d_B(b) \cdot d + b \cdot d_B(d)) \\
&= T(b) \cdot d_A(T(d)) + d_A(T(b)) \cdot T(d) - T^{(2n)}(d_B(b) \cdot d) - T^{(2n)}(b \cdot d_B(d)) \\
&= T(b) \cdot d_A(T(d)) + d_A(T(b)) \cdot T(d) - T^{(2n)}(d_B(b))T^{(2n)}(d) \\
&- T^{(2n)}(b)T^{(2n)}(d_B(d)) \\
&= T(b) \cdot d_A(T(d)) + d_A(T(b)) \cdot T(d) - T^{(2n)}(d_B(b)) \cdot T(d) \\
&- T(b) \cdot T^{(2n)}(d_B(d)) \\
&= T(b) \cdot (d_A(T(d)) - T^{(2n)}(d_B(d))) \\
&+ T(d) \cdot (d_A(T(b)) - T^{(2n)}(d_B(b))) \\
&= T(b) \cdot R(d) + T(d) \cdot R(b).
\end{aligned}$$

Also $d_A(T(b)) \cdot a = R(b) \cdot a + T^{(2n)}(d_B(b)) \cdot a \ \ (a \in A, b \in B)$, and so by Proposition 3.2 D is a derivation. Since $A \times_T B$ is $(2n)-$weakly amenable, there exists $(F, G) \in (A \times_T B)^{(2n)}$ such that $D = \delta_{(F,G)}$. Hence

$$\begin{aligned}
D((a, b)) &= (d_A(a) + R(b), d_B(b)) = (a, b) \cdot (F, G) - (F, G) \cdot (a, b) \\
&= ((a + T(b)) \cdot F + T^{(2n)}(G) \cdot a, b \cdot G) \\
&- ((a + T(b)) \cdot F + T^{(2n)}(G) \cdot a, G \cdot b) \\
&= (0, b \cdot G - G \cdot b) = (0, \delta_G(b)) \ \ (a \in A, b \in B).
\end{aligned}$$

It follows that $d_A = 0$ and $d_B = \delta_G$. So A and B are $(2n)-$weakly amenable.
For the converse let A and B be $(2n)-$weakly amenable and let
$D : A \times_T B \to (A \times_T B)^{(2n)}$ be a derivation. By Proposition 3.2 there exist derivation d_B and bounded linear mappings d_A, R, S with their described properties mentioned in Proposition 3.2 such that
$D = (d_A + R, S + d_B)$. Since $S(ac) = 0\ (a, c \in A)$, the equality $\bar{A^2} = A$ implies that, $S = 0$. By part (ii) of Proposition 3.2, since $S = 0$ so d_A is a derivation.

As A is commutative and $(2n)-$weakly amenable, $d_A = 0$. On the other hand since d_B is a derivation, there exists $G \in B^{(2n)}$ such that $d_B = \delta_G$. By part (iv) of Proposition 3.2 we have $R(b) \cdot a + T^{(2n)}(d_B(b)) \cdot a = 0$. It follows that

$$\begin{aligned} 0 &= R(b) \cdot a + T^{(2n)}(b \cdot G - G \cdot b) \cdot a \\ &= R(b) \cdot a + (T^{(2n)}(b)T^{(2n)}(G) - T^{(2n)}(G)T^{(2n)}(b)) \cdot a \\ &= R(b) \cdot a + (T(b)T^{(2n)}(G) - T(b)T^{(2n)}(G)) \cdot a \\ &= R(b) \cdot a. \end{aligned}$$

Following part (v) of Proposition 3.2,

$$R(bd) = T(b) \cdot R(d) + T(d) \cdot R(b) = 0 \ (b, d \in B).$$

So the assumption, $\bar{B^2} = B$ implies that $R = 0$. So $D = \delta_{(0,G)}$. Hence $A \times_T B$ is $(2n)-$weakly amenable. □

Recall that an arbitrary Banach algebra A is permanently weakly amenable if A is $n-$weakly amenable for each $n \in \mathbb{N}$.

Corollary 3.5. *Let A be a commutative Banach algebra, B be a Banach algebra, and let $T : B \to A$ be an algebra morphism with $\|T\| \le 1$. Also Let $\bar{A^2} = A, \bar{B^2} = B$, then $A \times_T B$ is permanently weakly amenable if and only if both A and B are permanently weakly amenable.*

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