Chamchuri Journal of Mathematics
Volume 4(2012), 51–61
http://www.math.sc.chula.ac.th/cjm

Dual $L-$Almost Distributive Lattices

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Received 21 May 2012
Accepted 26 November 2012

Abstract: In this paper, the concept of a dual $L-$ Almost Distributive Lattice (Dual $L-$ADL) is introduced as a generalization of a dual $L-$ algebra. Different necessary and sufficient conditions for an ADL to become a dual $L-$ADL are derived. It is proved that every dual $L-$ADL is a dual Stone Almost Distributive Lattice as well as a dually Normal Almost Distributive Lattice. A dual $L-$ADL is characterized in terms of its principal ideals and prime ideals.

Keywords: Almost Distributive Lattice(ADL), Principal ideal, Dual Heyting Almost Distributive Lattice(Dual $H-$ADL), Dual $L-$ Algebra, Dual $L-$ Almost Distributive Lattice (Dual $L-$ADL)

2000 Mathematics Subject Classification: 06D99, 06D20

1 Introduction

Heyting algebra is a relatively pseudo-complemented distributive lattice which arises from non-classical logic and was first investigated by T. Skolem about 1920. It was named as Heyting algebra after the Dutch Mathematician Arend Heyting. It was introduced by G. Birkhoff [1] under a different name Brouwerian lattice and with a diffent notation. Later H. B. Curry [3] in 1963 developed the theory of Heyting algebras and G. Epsetin [4] developed the concept of a $L-$algebra. The concept of an Almost Distributive Lattice (ADL) was introduced by U.M. Swamy and G. C. Rao [11] as a common abstraction to most of the existing ring

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† *The author is supported by U.G.C. under XI Plan*

theoretic generalizations of a Boolean algebra on one hand and the distributive lattices on the other. Later, G. C. Rao, Berhanu and R. Prasad [5] introduced the concept of an $L-$ ADL and derived many important results. Unlike in lattices, the dual of an ADL in not an ADL, in general. For this reason, in this paper, we introduce the concept of a Dual $L-$ Almost Distributive Lattice (dual $L-$ADL) as a generalization of a dual $L-$ algebra. We derive a number of important results satisfied by a dual $L-$ADL. We also characterize a dual $L-$ADL in terms of its principal ideals and prime ideals.

2 Preliminaries

In this section, we give the necessary definitions and important properties of an ADL taken from [11] for ready reference.

Definition 2.1. [11] An algebra $(A, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice(ADL) if it satisfies the following axioms:

(i) $x \vee 0 = x$

(ii) $0 \wedge x = 0$

(iii) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

(iv) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(v) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(vi) $(x \vee y) \wedge y = y$ for all $x, y, z \in A$.

A non-empty subset I of an ADL A is called an ideal of A if $x \vee y \in I$ and $x \wedge a \in I$ for any $x, y \in I$ and $a \in A$. The principal ideal of A generated by x is denoted by $(x]$. The set $PI(A)$ of all principal ideals of A forms a distributive lattice under the operations $\vee, \wedge$ defined by $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \wedge y]$ in which $(0]$ is the least element. If A has a maximal element m, then $(m]$ is the greatest element of $PI(A)$.

In our paper [8], we introduced the concept of a dual Stone ADLs and studied its properties. Now we begin with the following definition which is taken from [7].

Definition 2.2. [7, 8] Let $(A, \vee, \wedge, 0)$ be an ADL with a maximal element m. Then a unary operation $x \longrightarrow x_*$ on A is called a dual pseudo-complementation on A if, for any $x, y \in A$, the following conditions are satisfied:

(i) $x \vee y = m \Longrightarrow (x_* \vee y) \wedge m = y \wedge m$.

(ii) $x \vee x_*$ is a maximal element in A.

(iii) $(x \wedge y)_* = x_* \vee y_*$.

An ADL A with a dual pseudo-complementation is called a dually Pseudo-Complemented Almost Distributive Lattice (or, simply a dual PCADL). If, in-addition, $x_* \wedge x_{**} = 0$ for all $x \in A$, then A is called a dual Stone Almost Distributive Lattice (or, simply a dual Stone ADL).

The following theorem gives the characterization of a dual Stone ADL.

Theorem 2.3. *[8] The following are equivalent:*

(i) A *is a dual Stone ADL.*

(ii) $x^+ \vee y^+ = A$ *whenever* $x, y \in A$ *and* $x \vee y$ *is a maximal.*

(iii) $x_* \wedge y_* = 0$ *whenever* $x, y \in A$ *and* $x \vee y$ *is a maximal.*

For other properties of a dual PCADL and a dual Stone ADL, we refer to [7] and [8].

Definition 2.4. [9] An ADL A is said to be a normal if for any $x, y \in A, x \wedge y = 0$, then $(x)^* \vee (y)^* = A$ where $(x)^* = \{\ y \in A \mid x \wedge y = 0\ \}$.

Unlike in lattices, the dual of an ADL is not an ADL, in general. For this reason, the G. C. Rao and S. Ravi Kumar [10] introduced the concept a dually Normal Almost Distributive Lattices as follows.

Definition 2.5. [10] An ADL A with maximal elements is said to be dually normal if for any $x, y \in A$, $x \vee y$ is a maximal, then $(x)^+ \vee (y)^+ = A$ where $(x)^+ = \{y \in A \mid x \vee y$ is a maximal element$\}$.

For other properties of a dually Normal Almost Distributive Lattices, we refer [10].

3 Dual $L-$ADLs

In [6], G.C. Rao, Berhanu and M.V. Ratna Mani extended the concept of a Heyting algebra to the class of ADLs and in [5], G.C. Rao, Berhanu and R. Prasad studied the properties of an $L-$ADLs. In our paper [12], we introduced the concept of a dual Heyting Almost Distributive Lattice as follows.

Definition 3.1. [12] Let $(A, \vee, \wedge, 0)$ be an ADL with a maximal element m. Suppose $\leftarrow$ is a binary operation on A satisfying the following conditions:

(i) $x \leftarrow x = 0$

(ii) $[x \vee (x \leftarrow y)] \wedge m = (x \vee y) \wedge m$

(iii) $[(x \leftarrow y) \vee y] \wedge m = y \wedge m$

(iv) $z \leftarrow (x \vee y) = (z \leftarrow x) \vee (z \leftarrow y)$

(v) $(x \wedge y) \leftarrow z = (x \leftarrow z) \vee (y \leftarrow z)$ for all $x, y, z \in A$.

Then $(A, \vee, \wedge, \leftarrow, 0, m)$ is called a dual Heyting Almost Distributive Lattice (or, simply a dual $H-$ADL).

If $(A, \vee, \wedge, \leftarrow, 0, m)$ is a dual $H-$ADL, then, for $x, y, z \in A$, we observed that $(x \vee z) \wedge m \geq y \wedge m$ if and only if $z \wedge m \geq (x \leftarrow y) \wedge m$.

Now we introduce the concept of a dual $L-$Almost Distributive Lattice as follows.

Definition 3.2. A dual $H-$ADL $(A, \vee, \wedge, \leftarrow, 0, m)$ is said to be a dual $L-$Almost Distributive Lattice(or, simply a dual $L-$ADL), if for any $x, y \in A$,
$(x \leftarrow y) \wedge (y \leftarrow x) = 0$.

Now we prove the following theorem.

Theorem 3.3. *Let A be an ADL with a maximal element m. Then, the following are equivalent:*

(i) A is a dual $L-$ADL.

(ii) $[0,\ a]$ is a dual $L-$algebra for all $a \in A$.

(iii) $[0,\ m]$ is a dual $L-$algebra.

Proof. (i)$\Longrightarrow$(ii): Suppose $(A, \vee, \wedge, \leftarrow, 0, m)$ is a dual $L-$ADL and $a \in A$. Define a binary operation $\leftarrow_a$ on $[0,\ a]$ by $x \leftarrow_a y = (x \leftarrow y) \wedge a$ for any $x,\ y \in [0,\ a]$. Then, by Theorem 3.4 [12], we get that $[0,\ a]$ is a dual $H-$ADL. For $x,\ y \in [0,\ a]$, $(x \leftarrow_a y) \wedge (y \leftarrow_a x) = (x \leftarrow y) \wedge a \wedge (y \leftarrow x) \wedge a = (x \leftarrow y) \wedge (y \leftarrow x) \wedge a = 0 \wedge a = 0$. Thus $[0,\ a]$ is a dual $L-$algebra. (ii)$\Longrightarrow$(iii) is trivial. (iii)$\Longrightarrow$(i): Suppose $([0,\ m], \vee, \wedge, \leftarrow_m, 0, m)$ is a dual $L-$algebra. For any $x, y \in A$, define $x \leftarrow y = (x \wedge m) \leftarrow_m (y \wedge m)$. Then, by Theorem 3.4 [12], we get that A is a dual $H-$ADL. For $x, y \in A$, $(x \leftarrow y) \wedge (y \leftarrow x) = [(x \wedge m) \leftarrow_m (y \wedge m)] \wedge [(y \wedge m) \leftarrow_m (x \wedge m)] = 0 \wedge m = 0$. Hence A is a dual $L-$ADL. □

Here after-wards, A stands for a dual $L-$ADL with a maximal element m unless otherwise stated.

Theorem 3.4. *Let $(A, \vee, \wedge, \leftarrow, 0, m)$ be a dual $L-$ADL. Then, for any maximal element n in A, $(A, \vee, \wedge, \leftarrow_n, 0, n)$ is a dual $L-$ADL where $x \leftarrow_n y = (x \leftarrow y) \wedge n$ for any $x, y \in A$.*

Proof. Let $(A, \vee, \wedge, \leftarrow, 0, m)$ be a dual $L-$ADL. For any maximal element $n \in A$, define $x \leftarrow_n y = (x \leftarrow y) \wedge n$ for any $x, y \in A$. Then, by Theorem 3.8 [12], $(A, \vee, \wedge, \leftarrow_n, 0, n)$ is a dual $H-$ADL. For any $x, y \in A$,

$$\begin{aligned}(x \leftarrow_n y) \wedge (y \leftarrow_n x) &= [(x \leftarrow y) \wedge n] \wedge [(y \leftarrow x) \wedge n] \\ &= [(x \leftarrow y) \wedge (y \leftarrow x)] \wedge m \wedge n \\ &= 0 \wedge m \wedge n \\ &= 0.\end{aligned}$$

Hence $(A, \vee, \wedge, \leftarrow_n, 0, n)$ is a dual $L-$ADL. □

The following result taken from [12] will be useful to prove the next results.

Lemma 3.5. *[12] Let A be a dual $H-$ADL with a maximal element m. Then, for any $x,\ y,\ z \in A$, we have the following:*

(i) $(0 \leftarrow x) \wedge m = x \wedge m$.

(ii) *If $x \leq y$, then $z \leftarrow x \leq z \leftarrow y$ and $x \leftarrow z \geq y \leftarrow z$.*

(iii) $[(x \vee y) \leftarrow z] \wedge m = [(x \leftarrow y) \leftarrow z)] \wedge m$.

Now we prove the following theorem.

Theorem 3.6. *Let A be a dual $H-ADL$ with a maximal element m. Then A is a dual $L-ADL$ if and only if, for any $x, y, z \in A$,*
either (i) $(z \leftarrow (x \wedge y)) \wedge m = (z \leftarrow x) \wedge (z \leftarrow y) \wedge m$
or (ii) $((x \vee y) \leftarrow z) \wedge m = (x \leftarrow z) \wedge (y \leftarrow z) \wedge m$
holds.

Proof. Let A be a dual $H-$ADL with a maximal element m and $x, y, z \in A$. Suppose (i) condition holds. For $x, y \in A$,

$$\begin{aligned} 0 = 0 \wedge m &= ((x \wedge y) \leftarrow (x \wedge y)) \wedge m \\ &= [(x \wedge y) \leftarrow x] \wedge [(x \wedge y) \leftarrow y] \wedge m \\ &= [(x \leftarrow x) \vee (y \leftarrow x)] \wedge [(x \leftarrow y) \vee (y \leftarrow y)] \wedge m \\ &= (x \leftarrow y) \wedge (y \leftarrow x) \wedge m. \end{aligned}$$

Hence $(x \leftarrow y) \wedge (y \leftarrow x) = 0$. Therefore A is a dual $L-$ADL. Similarly, we get that A is a dual $L-$ADL when (ii) holds. Conversely, suppose A is a dual $L-$ADL and $x, y \in A$. Since $(x \vee y) \wedge m \geq x \wedge m$, we get $x \leftarrow z \geq (x \vee y) \leftarrow z$. Similarly, we get $y \leftarrow z \geq (x \vee y) \leftarrow z$ and hence $(x \leftarrow z) \wedge (y \leftarrow z) \wedge m \geq ((x \vee y) \leftarrow z) \wedge m$. Let $p = (x \vee y) \leftarrow z$. Then $[(x \vee y) \vee p] \wedge m \geq z \wedge m$. Now,

$$\begin{aligned} &((x \vee y) \leftarrow z) \wedge m \\ &\quad = (0 \vee p) \wedge m \\ &\quad = \{[(x \leftarrow y) \wedge (y \leftarrow x)] \vee p\} \wedge m \\ &\quad = [p \vee (x \leftarrow (x \vee y))] \wedge [p \vee (y \leftarrow (y \vee x))] \wedge m \\ &\quad = [(p \vee (x \leftarrow p)) \vee (x \leftarrow (x \vee y))] \wedge [p \vee (y \leftarrow p) \vee (y \leftarrow (x \vee y))] \wedge m \\ &\quad \geq (x \leftarrow (x \vee y \vee p)) \wedge (y \leftarrow (x \vee y \vee p)) \wedge m \\ &\quad \geq (x \leftarrow z) \wedge (y \leftarrow z) \wedge m. \end{aligned}$$

Hence $((x \vee y) \leftarrow z) \wedge m = (x \leftarrow z) \wedge (y \leftarrow z) \wedge m$. Thus we get (i).
Again, since $x \wedge m \geq (x \wedge y) \wedge m$ and $y \wedge m \geq (x \wedge y) \wedge m$, we get $(z \leftarrow x) \wedge (z \leftarrow y) \wedge m \geq (z \leftarrow (x \wedge y)) \wedge m$. If we take $s = z \leftarrow (x \wedge y)$, then $(z \vee s) \wedge m \geq (x \wedge y) \wedge m$. Now,

$$
\begin{aligned}
(z \leftarrow (x \wedge y)) \wedge m &= (0 \vee s) \wedge m \\
&= \{[(x \leftarrow y) \wedge (y \leftarrow x)] \vee s\} \wedge m \\
&= [s \vee ((x \wedge y) \leftarrow y)] \wedge [s \vee ((x \wedge y) \leftarrow x)] \wedge m \\
&\geq (s \vee ((s \vee z) \leftarrow y)) \wedge (s \vee ((s \vee z) \leftarrow x)) \wedge m \\
&= (s \vee (s \leftarrow (z \leftarrow y))) \wedge (s \vee (s \leftarrow (z \leftarrow x))) \wedge m \\
&\geq (z \leftarrow y) \wedge (z \leftarrow x) \wedge m.
\end{aligned}
$$

Hence we get the result. □

Theorem 3.7. *For* $x, y \in A$, $[((x \leftarrow y) \leftarrow y) \vee ((y \leftarrow x) \leftarrow x)] \wedge m = x \wedge y \wedge m$.

Proof. Let $x, y \in A$. Then $[y \vee (y \leftarrow x)] \geq x \wedge m$, and hence $y \wedge m \geq [(y \leftarrow x) \leftarrow x] \wedge m$. Again, since $[(x \leftarrow y) \vee y] \wedge m = y \wedge m$, we get $y \wedge m \geq [(x \leftarrow y) \leftarrow y] \wedge m$, and hence $y \wedge m \geq [((x \leftarrow y) \leftarrow y) \vee ((y \leftarrow x) \leftarrow x)] \wedge m$. Interchanging x and y, we get $x \wedge m \geq [((y \leftarrow x) \leftarrow x) \vee ((x \leftarrow y) \leftarrow y)] \wedge m$. Hence $(x \wedge y) \wedge m \geq [((x \leftarrow y) \leftarrow y) \vee ((y \leftarrow x) \leftarrow x)] \wedge m$. On the other hand, since $x \wedge m \geq (x \wedge y) \wedge m$, we get $((y \leftarrow x) \leftarrow x) \wedge m \geq [(y \leftarrow x) \leftarrow (x \wedge y)] \wedge m$. Now

$$
\begin{aligned}
&[((x \leftarrow y) \leftarrow y) \vee ((y \leftarrow x) \leftarrow x)] \wedge m \\
&\qquad \geq [((x \leftarrow y) \leftarrow (x \wedge y)) \vee ((y \leftarrow x) \leftarrow (x \wedge y))] \wedge m \\
&\qquad = [((x \leftarrow y) \wedge (y \leftarrow x)) \leftarrow (x \wedge y)] \wedge m \\
&\qquad = (0 \leftarrow (x \wedge y)) \wedge m \\
&\qquad = x \wedge y \wedge m.
\end{aligned}
$$

Hence we get $x \wedge y \wedge m = [((x \leftarrow y) \leftarrow y) \vee ((y \leftarrow x) \leftarrow x)] \wedge m$. □

If $(A, \vee, \wedge, \leftarrow, 0, m)$ is a dual $H-$ADL and if we define for any $x \in A$, $x_* = x \leftarrow m$, then $(A, \vee, \wedge, _*, 0, m)$ is a dual PCADL [12]. Now we prove the following.

Theorem 3.8. *A dual* $L-$*ADL* A *with a maximal element* m *is a dual Stone ADL as well as a dually normal ADL.*

Proof. For each $x \in A$, define $x_* = (x \leftarrow m)$. Then, by Theorem 3.17[12], A is a dual PCADL. Now, for any $x \in A$, $x_* \wedge x_{**} \wedge m = (x \leftarrow m) \wedge (x_* \leftarrow m) \wedge m = ((x \vee x_*) \leftarrow m) \wedge m = [((x \vee x_*) \wedge m) \leftarrow m] \wedge m = (m \leftarrow m) \wedge m = 0$. Hence $x_* \wedge x_{**} = 0$. Therefore A is a dual Stone ADL. By Theorem 2.3, we get that A is a dually normal ADL. □

Theorem 3.9. *For any* $x, y \in A$, $(x \leftarrow y)_* = (x_* \wedge y)_*$.

Proof. Let $x, y \in A$. Since $(x \vee (x_* \wedge y)) \wedge m = (x \vee x_*) \wedge (x \vee y) \wedge m = (x \vee y) \wedge m \geq y \wedge m$, we get $(x_* \wedge y) \wedge m \geq (x \leftarrow y) \wedge m$ and hence $(x \leftarrow y)_* \geq (x_* \wedge y)_*$. On the other hand,

$$
\begin{aligned}
(x \vee (x \leftarrow y)) \wedge m \geq y \wedge m \\
&\Longrightarrow x_* \wedge (x \vee (x \leftarrow y)) \wedge m \geq x_* \wedge y \wedge m \\
&\Longrightarrow ((x_* \wedge x) \vee (x_* \wedge (x \leftarrow y)) \wedge m \geq x_* \wedge y \wedge m \\
&\Longrightarrow ((x_* \wedge x) \vee (x \leftarrow y)) \wedge m \geq x_* \wedge y \wedge m \\
&\Longrightarrow [((x_* \wedge x) \vee (x \leftarrow y)) \wedge m]_* \leq (x_* \wedge y \wedge m)_* \\
&\Longrightarrow (x_* \wedge y)_* \geq (x_{**} \vee x_*) \wedge (x \leftarrow y)_* = (x_{**} \vee x_*) \wedge m \wedge (x \leftarrow y)_* \\
&\Longrightarrow (x_* \wedge y)_* \geq (x \leftarrow y)_*.
\end{aligned}
$$

Hence we get $(x \leftarrow y)_* = (x_* \wedge y)_*$. ☐

A dual $L-$ADL becomes a dual $L-$algebra once it becomes a lattice. Therefore we get a number of equivalent conditions for a dual $L-$ADL to become a dual $L-$algebra as a consequence of Theorem 1.13 [11].

Theorem 3.10. *Let A be a dual $L-$ ADL with a maximal element m. The following are equivalent:*

(i) A *is a distributive lattice.*

(ii) A *is a* $L-$ *algebra.*

(iii) A *is a Heyting algebra.*

(iv) $(A, \leq)$ *is directed above.*

(v) $\vee$ *is commutative in* A*.*

(vi) $\wedge$ *is commutative in* A*.*

(vii) $\vee$ *is right distributive over* $\wedge$ *in* A*.*

(viii) $A = \{(a, b) \in A \times A \mid b \wedge a = a\}$ *is anti-symmetric.*

(ix) *For any* $x, y, z \in A$, $x \vee z \geq y \Longleftrightarrow z \geq (x \leftarrow y)$.

In the following theorem, we characterize a dual $L-$ADL in terms of prime ideals.

Theorem 3.11. *If A is a dual $H-$ADL with a maximal element m, then A is a dual $L-$ADL if and only if, for any prime ideal P of A, the set of prime ideals containing it is linearly ordered.*

Proof. Suppose A is a dual $L-$ADL with a maximal element m and P is a prime ideal of A. Let I and J be prime ideals containing P. Suppose that I and J are incomparable. Then there exist $x \in I/J$ and $y \in J/I$. Since A is a dual $L-$ADL, we get $(x \leftarrow y) \wedge (y \rightarrow x) = 0$. But P is a prime ideal of A, then either $(x \leftarrow y) \in P$ or $(y \leftarrow x) \in P$. Suppose that $(x \leftarrow y) \in P$. Since $[x \vee (x \leftarrow y)] \wedge m \geq y \wedge m$ and $y \wedge m \notin I$, we get $[x \vee (x \leftarrow y)] \wedge m \notin I$. But $x \in I$ and $(x \leftarrow y) \wedge m \in G$. Then $[x \vee (x \leftarrow y)] \wedge m \in I$ which is contradiction. Therefore I and J are comparable. Hence, the set of prime ideals containing P is linearly ordered. Conversely, suppose that for any prime ideal P of A, the set of prime ideals containing it is linearly ordered. Suppose, for any $x, y \in A$, $\{(x \leftarrow y) \wedge (y \leftarrow x)\} \neq 0$. Then there exists a prime ideal Q of A such that $(x \leftarrow y) \wedge (y \rightarrow x) = 0 \notin Q$. Let P_1 be the ideal generated by P and $(x \leftarrow y)$ and P_2 be the ideal generated by F and $(y \leftarrow x)$. Suppose that $(y \leftarrow x) \in P_1$. Then $(y \leftarrow x) \wedge m \leq z \wedge (x \rightarrow y)$ for some $z \in P$. Now

$$\begin{aligned} y \wedge m \geq (x \leftarrow y) \wedge m &\Longrightarrow (z \vee y) \wedge m \geq [z \vee (x \leftarrow y)] \wedge m \geq (y \leftarrow x) \wedge m \\ &\Longrightarrow (y \vee z) \wedge m \geq [y \vee (y \leftarrow x)] \geq x \wedge m \\ &\Longrightarrow (z \vee y) \wedge m \geq x \wedge m \\ &\Longrightarrow z \wedge m \geq (y \leftarrow x) \wedge m. \end{aligned}$$

Since $z \in P$, we get $(y \leftarrow x) \wedge m \in P$ which is a contradiction. Therefore $(y \leftarrow x) \wedge m \notin P_1$. Similarly, we can prove that $(x \leftarrow y) \wedge m \notin P_2$. Therefore there exists a prime ideal I containing P_1 but not $(y \leftarrow x) \wedge m$ and a prime filter J containing P_2 but not $(x \leftarrow y) \wedge m$. Hence I and J are incomparable which is contradiction. Hence $(x \leftarrow y) \wedge (y \leftarrow x) = 0$. Therefore A is a dual $L-$ADL. □

Finally, we conclude this paper with the following theorem.

Theorem 3.12. *Let A be an ADL with a maximal element m. Then A is a dual $L-$ADL if and only if $PI(A)$ is a dual $L-$algebra.*

Proof. Suppose A is a dual $L-$ADL. For $x, y \in A$, if we define $(x] \leftarrow (y] = (x \leftarrow y]$. Then, by Theorem 3.13 [12], PI(A) is a dual Heyting algebra. Now we prove that PI(A) is a dual $L-$ algebra. For $x, y \in A$, $\{(x] \leftarrow (y]\} \cap \{(y] \leftarrow (x]\} = (x \leftarrow y] \cap (y \leftarrow x] = ((x \leftarrow y) \wedge (y \leftarrow x)] = (0]$ and hence PI(A) is a dual $L-$algebra. Conversely, suppose PI(A) is a dual $L-$algebra. For $x, y \in A$, if we define $x \leftarrow y = c \wedge m$ where $(x] \leftarrow (y] = (c]$, $c \in A$. Then, by Theorem 3.13 [12], A is a dual $H-$ADL. Now, we prove that A is a dual $L-$ADL. Let $x, y, \in A$ and suppose $(x] \leftarrow (y] = (s]$ and $(y] \leftarrow (x] = (t]$. Then $x \leftarrow y = s \wedge m$ and $y \leftarrow x = t \wedge m$. Since $(x] \leftarrow (y] \cap (y] \leftarrow (x] = (0]$, we get $(s] \cap (t] = (0]$ and hence $s \wedge t = 0$. Now $(x \leftarrow y) \wedge (y \leftarrow x) = s \wedge m \wedge t \wedge m = s \wedge t \wedge m = 0$. Hence A is a dual $L-$ADL. □

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