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Endo-Regularity of Generalized Wheel Graphs

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Abstract: A graph G is endo-regular (endo-orthodox, endo-completely-regular) if the monoid of all endomorphisms on G is regular (orthodox, completely regular respectively). In this paper, we characterize endo-regular (endo-orthodox, endo-completely-regular) of generalized wheel graphs $W_n(m)$. For each $m \geq 2$, we found that the $W_n(m)$ is endo-regular (endo-orthodox resp.) if and only if n is odd and $m = 2$ and $W_n(m)$ is endo-completely-regular if and only if it is $W_3(2)$.

Keywords: generalized wheel graph, endomorphism, regular, orthodox, completely regular

2000 Mathematics Subject Classification: 05C25; 05C38

1 Introduction and Preliminaries

In [3], W. Li characterized regular endomorphisms on arbitrary graphs. The characterizations of endo-regular and endo-orthodox connected bipartite graphs were explicitly found in [10] and [1], respectively. A characterization of endo-regularity of paths and cycles was found in [7]. In [8, 9], N. Pipattanajinda, J. Thamkeaw and Sr. Arworn characterized endo-regularity of cycle book graphs.

As usual we denote by $V(G)$ and $E(G)$ the *vertex set* and the *edge set* of the *graph* G, respectively. Let G and H be two simple graphs. The *union* of G and

H, denoted by $G \cup H$, is a graph such that the vertex set $V(G \cup H) = V(G) \cup V(H)$ and the edge set $E(G \cup H) = E(G) \cup E(H)$. The *join* of G and H, denoted by $G + H$, is a graph such that the vertex set $V(G + H) = V(G) \cup V(H)$ and the edge set $E(G + H) = E(G) \cup E(H) \cup \{\{u, v\} | u \in V(G), v \in V(H)\}$.

A (*graph*) *homomorphism* from a graph G to a graph H is a mapping $f : V(G) \to V(H)$ which preserves edges, i.e. $\forall u, v \in V(G)$, $\{u, v\} \in E(G)$ implies $\{f(u), f(v)\} \in E(H)$. A homomorphism f is an *isomorphism* if f is bijective and f^{-1} is also a homomorphism. A homomorphism (resp. isomorphism) f from G to itself is called an *endomorphism* (resp. *automorphism*) of G. Denoted the class of all endomorphisms and of all automorphisms of G by $End(G)$ and $Aut(G)$, respectively. It is well known that for any graph G, $End(G)$ with the composition forms a monoid when $Aut(G)$ forms a group.

The graph with the vertex set $\{0, 1, \dots, n\}$ and the edge set $\{\{i, i+1\} | i = 0, 1, \dots, n-1\}$ is called a *path* P_n of length n. The graph with the vertex set $\{1, 2, \dots, n\}$, such that $n \geq 3$ and the edge set $\{\{i, i+1\} | i = 1, 2, \dots, n\}$ (with addition modulo n) is called a *cycle* C_n of length n. We denote by $Hom(G, H)$ the class of all homomorphisms from a graph G to a graph H, and denote by $Hom^i_j(P_m, P_n)$ the class of all homomorphisms $f \in Hom(P_m, P_n)$, such that $f(0) = i$ and $f(m) = j$. Then

Lemma 1.1. *Let m, n be even integers, and $f \in Hom(P_m, P_n)$. If $f(0)$ is even (odd resp.), then $f(m)$ is also even (odd resp.).*

Corollary 1.2. *If n is even, then*

$$|Hom^0_1(P_n, P_2)| = |Hom^1_0(P_n, P_2)| = |Hom^1_2(P_n, P_2)| = |Hom^2_1(P_n, P_2)| = 0.$$

Lemma 1.3. *If m, n are positive integers, $m \geq 3$, then*

$$|Hom(C_m, P_n)| = \sum_{i=0}^{n-1} [|Hom^i_{i-1}(P_{m-1}, P_n)| + |Hom^i_{i+1}(P_{m-1}, P_n)|].$$

Corollary 1.4. *If m is odd, then $|Hom(C_m, P_2)| = 0$.*

A factor graph I_f of G under f which is a subgraph of G is called the *endomorphic image* of G under f. This means, $V(I_f) = f(V(G))$ and $\{f(u), f(v)\} \in E(I_f)$ if and only if there exist $u' \in f^{-1}f(u)$ and $v' \in f^{-1}f(v)$ such that $\{u', v'\} \in E(G)$, where $f^{-1}(t)$ denotes the set of *preimages* of some vertex t of

G under the mapping f. By ρ_f, we denote the *equivalence relation* on $V(G)$ induced by f, i. e. for any $u, v \in V(G)$, $(u, v) \in \rho_f$ if and only if $f(u) = f(v)$.

Let S be a semigroup (monoid resp.). An element a of S is called an *idempotent* if $a^2 = a$. An element a of S is called a *regular* if $a = aa'a$ for some $a' \in S$, such a' is called a *pseudo inverse* to a. The semigroup S is called *regular* if every element of S is regular. A regular element a of S is called *completely regular* if there exists a pseudo inverse a' to a such that $aa' = a'a$. In this case we call a' a *commuting pseudo inverse* to a. The semigroup S is called *completely regular* if every element of S is completely regular. A regular semigroup S is called *orthodox* if the set of all idempotent elements of S (denoted by $Idpt(S)$) forms a semigroup under the operation of S. The Green's relations $\mathcal{H}$ on S are defined by $a\mathcal{H}b \Leftrightarrow S^1a = S^1b$ and $aS^1 = bS^1$. Denote the equivalence class $\mathcal{H}$ of S containing element a by H_a.

Note that every bijective endomorphism on a finite graph is an automorphism, then it is regular.

Lemma 1.5. [6] *A semigroup S is completely regular if and only if S is a union of (disjoint) groups.*

Lemma 1.6. [6] *Let S be a semigroup and e is an idempotent of S. Then H_e is a subgroup of S.*

Lemma 1.7. [5] *Let G be a graph. Suppose $f, g \in End(G)$ and f, g are regular. Then $f\mathcal{H}g$ if and only if $\rho_f = \rho_g$ and $I_f = I_g$.*

We call a graph G *endo-regular* (*endo-orthodox*, *endo-completely-regular*, *unretractive*), if the monoid $End(G)$ is regular (orthodox, completely regular, group resp.). Note that for any cycle C_n is unretractive if and only if n is odd, and every complete graph of n vertices, K_n is unretractive. The following lemmas are useful for this paper.

Lemma 1.8. [2] *Let G and H be graphs. The $G + H$ is unretractive if and only if G and H are unretractive.*

Lemma 1.9. [7] *A cycle C_n is endo-regular if and only if n is odd, or n is $4, 6$, or 8.*

Lemma 1.10. [4] *Let G be a graph. Then G is endo-regular if and only if $G + K_n$ is endo-regular for any $n \geq 1$.*

Lemma 1.11. [1] *Let* G *be a bipartite graph. Then* G *is endo-orthodox if and only if* G *is one of the following graphs*: $K_1, K_2, P_2, P_3, C_4, 2K_1$ *and* $K_1 \cup K_2$.

Lemma 1.12. [7] *For all positive integer* n, *the cycle* C_{2n} *is not endo-completely-regular.*

An endomorphism f of G is called a *path strong* (*cycle strong*) endomorphism if every path (cycle resp.) $f(y_0), f(y_1), \ldots, f(y_l)$ of length l in $f(G)$, there exists $x_i \in f^{-1}f(y_i)$, for each $i = 0, 1, \ldots, l$ such that $x_0, x_1, \ldots, x_l$ is a path (cycle resp.) of length l in G. Denoted the class of all path strong and cycle strong endomorphisms of G by $pEnd(G)$ and $cEnd(G)$, respectively.

Lemma 1.13. *Let* G *be a graph and* $f \in End(G)$. *If* f *is regular, then* $f \in pEnd(G) \cap cEnd(G)$.

Proof. Let $f \in End(G)$ be a regular endomorphism on G, then $f = fgf$ for some $g \in End(G)$. Let $f(y_0), f(y_1), \ldots, f(y_l)$ be a path (cycle) of length l in $f(G)$. For any $i = 0, 1, \ldots, l$, let $x_i = gf(y_i)$. Then

1. $x_i \in f^{-1}f(y_i)$ because $f(x_i) = fgf(y_i) = f(y_i)$.
2. $\{x_i, x_{i+1}\} \in E(G)$ for all $i = 0, 1, \ldots, l-1$ because $g \in End(G)$ and $\{f(y_i), f(y_{i+1})\} \in E(G)$, imply that $\{gf(y_i), gf(y_{i+1})\} \in E(G)$.
3. $x_i \neq x_j$ *if* $i \neq j$, because if $i \neq j$ but $x_i = x_j$, then $f(y_i) = fgf(y_i) = f(x_i) = f(x_j) = fgf(y_j) = f(y_j)$ which is impossible.

Thus, $x_0, x_1, \ldots, x_l$ is the path (cycle resp.) of length l in G. □

2 Endo-Regular of Generalized Wheel Graphs

Let m, n be positive integers, $n \geq 3$. For each $i = 1, 2, \ldots, m$, let G_i be a graph which is isomorphic to the cycle C_n with the following vertex set $V(G_i) = \{1_i, 2_i, \ldots, n_i\}$, and edge set $E(G_i) = \{\{k_i, (k+1)_i\} | k = 1, 2, \ldots, n\}$ where $+$ is the addition modulo n.

A generalized wheel graph of m rounds, $W_n(m)$ is the graph which the vertex set and the edge set are

$$V(W_n(m)) = \bigcup_{i=1}^{m} V(G_i) \cup \{0\},$$

and

$$E(W_n(m)) = \bigcup_{k=1}^{n}\{0, k_1\} \cup \bigcup_{i=1}^{m} E(G_i) \cup \bigcup_{i=1}^{m-1} \{\{k_i, k_{i+1}\} | k = 1, 2, \ldots, n\},$$

respectively. For example of generalized wheel graph $W_5(3)$, see Fig. 1. (Note that $W_n(1)$ is a wheel graph which was denoted by W_n).

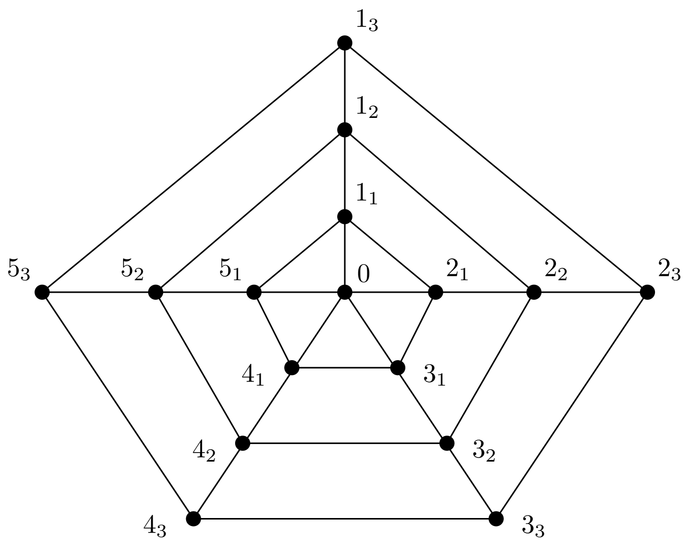

Fig. 1. Generalized wheel graph $W_5(3)$.

Since $W_n \cong C_n + K_1$ by Lemmas 1.8 - 1.12, it is easy to see that

Corollary 2.1. *For any wheel graph W_n.*

1. *W_n is unretractive if and only if n is odd.*
2. *A retractive wheel graph W_n is endo-regular if and only if $n = 4, 6, 8$.*
3. *A retractive wheel graph W_n is endo-orthodox if and only if $n = 4$.*
4. *Every retractive wheel graph W_n is not endo-completely-regular.*

For other generalized wheel graph $W_n(m), m \geq 2$

Lemma 2.2. *A generalized wheel graph $W_n(m)$ is retractive.*

Proof. Let $f : V(W_n(m)) \to V(W_n(m))$ be defined by $f(0) = 0$, and $f(x_i) = (x + i)_1$ for all $x_i \in V(W_n(m))$. Then $f \in End(W_n(m))$ is non-injective. □

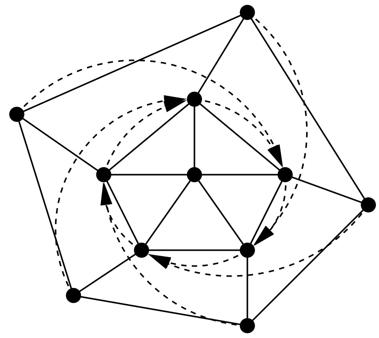

Fig. 2. A non-injective in $End(W_5(2))$.

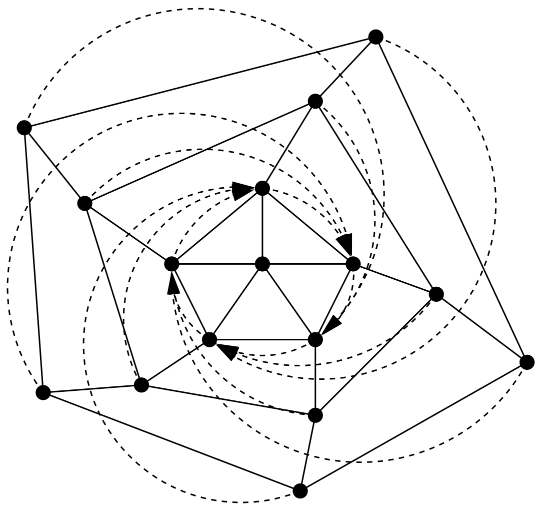

Fig. 3. A non-injective in $End(W_5(3))$.

Lemma 2.3. *For every* $f \in End(W_n(m))$, *either* $f(0) = 0$, *or* $f(0) \in V(G_1)$.

Proof. Let $f \in End(W_n(m))$. Since the induced subgraph by $\{0, 1_1, 2_1\}$ of $W_n(m)$ is isomorphic to cycle C_3, $\{f(0), f(1_1), f(2_1)\}$ must be a closed walk in $W_n(m)$, i.e. $\{f(0), f(1_1), f(2_1)\} = \{0, k_1, (k+1)_1\}$ for some $k = 1, 2, \ldots, n$. Therefore, $f(0) \in V(G_1) \cup \{0\}$. □

Let $f \in End(W_n(m))$. Denote the set of all elements $f(x)$ where $x \in V(G_1)$ by $f(G_1)$, and the restriction of f on G_1 by $f|_{G_1}$.

Lemma 2.4. *Let m, n be positive integers where n odd with $n > 3$. Then for all $f \in End(W_n(m))$, $f(0) = 0$, and $f(G_1) = V(G_1)$.*

Proof. Let n be an odd integer with $n > 3$ and $f \in End(W_n(m))$. By Lemma 2.3, $f(0) = 0$, or $f(0) \in V(G_1)$. Suppose $f(0) = k_1$ for some $k = 1, 2, \ldots, n$, then $f(y_1) \in \{0, (k-1)_1, (k+1)_1, k_2\}$ for all $y = 1, 2, \ldots, n$. But $\{f(0), f(y_1), f((y+1)_1)\}$ must be a closed walk, which is impossible if there exists $y_1 \in V(G_1)$ such that $f(y_1) = k_2$. Therefore, $f(G_1) \subseteq \{0, (k-1)_1, (k+1)_1\}$, i.e. $f|_{G_1} \in Hom(G_1, P)$ where P is the induced subgraph of $W_n(m)$ with $V(P) = \{0, (k-1)_1, (k+1)_1\}$ Since G_1 is isomorphic to C_n, and P is isomorphic to P_2, by Corolary1.4 $|Hom(G_1, P)| = |Hom(C_n, P_2)| = 0$ because $n-1$ is even. This contradicts to $f|_{G_1} \in Hom(G_1, P)$. Hence $f(0) = 0$.

Moreover, $f|_{G_1} \in End(G_1)$ and $End(G_1) \cong End(C_n)$ which is a group. Therefore, $f|_{G_1}$ is one to one, i.e. $f(G_1) = V(G_1)$. □

Lemma 2.5. *For any positive integer m, there exists $f \in End(W_3(m))$ such that $f(0) \neq 0$.*

Proof. Let $f : V(W_3(m)) \to V(W_3(m))$ be defined by $f(0) = 3_1$, $f(1_1) = 0$, and for each $x_i \neq 1_1$,

$$f(x_i) = \begin{cases} (x-1)_i, & i = 1; \\ x_{i-1}, & i \neq 1. \end{cases}$$

Then $f \in End(W_3(m))$ such that $f(0) \neq 0$. □

Lemma 2.6. *Let m, n be positive integers. If $m \geq 3$, then $W_n(m)$ is not endo-regular.*

Proof. Let m, n be positive integers, $m \geq 3$. We will show that there exists $f \in End(W_n(m))$ such that f is not regular. Let $f \in End(W_n(m))$ be defined by $f(0) = 0$, and for each $x_i \in V(W_n(m))$,

$$f(x_i) = \begin{cases} (x-1)_1, & i = 1; \\ x_{i-1}, & i > 1. \end{cases}$$

Thus $0, 1_1, 1_2$ is a path in $f(W_n(3))$. Since $f^{-1}(0) = \{0\}$, $f^{-1}(1_1) = \{1_2, 2_1\}$, and $f^{-1}(1_2) = \{1_3\}$, $W_n(3)$ has no path $0, x, 1_3$, where $x \in f^{-1}(1_1)$. From Lemma 1.13, f is not regular. □

Lemma 2.7. *The generalized wheel graph $W_n(2)$ is not endo-regular for all even positive integer n which $n > 3$.*

Proof. Let n be even positive integer, $n > 3$. Define $f \in End(W_n(2))$ by $f(0) = n_1$, and for each positive integer x; $0 < x \leq n$,

$$f(x_1) = \begin{cases} 0, & x \text{ is odd;} \\ 1_1, & x \text{ is even.} \end{cases}$$

and

$$f(x_2) = \begin{cases} 1_1, & x \text{ is odd;} \\ 2_1, & x \text{ is even.} \end{cases}$$

Thus $0, 2_1$ is a path in $f(W_n(2))$. Since $f^{-1}(0) = \{1_1, 3_1, 5_1, \ldots, (n-1)_1\}$, and $f^{-1}(2_1) = \{2_2, 4_2, \ldots, n_2\}$, $W_n(2)$ has no path x_1, y_2, where $x_1 \in f^{-1}(0)$ and $y_2 \in f^{-1}(2_1)$. From Lemma 1.13, f is not regular. □

Lemma 2.8. *Let $f \in End(W_3(2))$. If $f(x_2) = y_2$ for some $x_2, y_2 \in V(G_2)$, then f is bijective.*

Proof. Let $f \in End(W_3(2))$ and $f(x_2) = y_2$ for some $x_2, y_2 \in V(G_2)$. Since the induced subgraph by $\{1_2, 2_2, 3_2\}$ is isomorphic to cycle C_3, this implies that the induced by $\{f(1_2), f(2_2), f(3_2)\}$ is also isomorphic to cycle C_3. Thus $f(G_2) = V(G_2)$, implies $f(G_1) = V(G_1)$, and $f(0) = 0$. Therefore, f is bijective. □

Lemma 2.9. *Let $f \in End(W_3(2))$. If $f(x_2) \in V(G_1) \cup \{0\}$ for all $x_2 \in V(G_2)$, then f is regular.*

Proof. Let $f \in End(W_3(2))$ and $f(x_2) \in V(G_1) \cup \{0\}$ for all $x_2 \in V(G_2)$. Let $f_1 : V(W_3(1)) \to V(W_3(1))$ be such that $f_1(0) = f(0)$ and $f_1(x_1) = f(x_1)$ for all $x \in \{1, 2, 3\}$. Thus $f_1 \in End(W_3(1))$. Since $W_3(1) \cong K_4$, $End(W_3(1)) = Aut(W_3(1))$ and f_1 is bijective. Let $g \in End(W_3(2))$ be defined by $g(0) = f_1^{-1}(0)$, $g(x_1) = f_1^{-1}(x_1)$, and $g(x_2) = g((x+1)_1)$ for all $x = 1, 2, 3$. We can see that $f = fgf$. □

From Lemma 2.8 and Lemma 2.9, then

Corollary 2.10. *A generalized wheel graph $W_3(2)$ is endo-regular.*

Lemma 2.11. *A generalized wheel graph $W_n(2)$ is endo-regular, for all odd positive integer n, $n \geq 3$.*

Proof. From Corollary 2.10, $W_3(2)$ is endo-regular. Consider $W_n(2)$ when $n > 3$. By Lemma 2.4, $f(0) = 0$, $f(G_1) = V(G_1)$ and $f|_{G_1}$ is 1-1. Without loss of generality, for each $i_1 \in V(G_1)$ let $f(i_1) = (x+i-1)_1$ for some $x = 1, \ldots, n$. For each $i_2 \in V(G_2), f(i_2) \in \{0, (x+i)_1, (x+i-2)_1, (x+i-1)_2\}$. In any case of f, let $g \in End(W_n(2))$ be defined by $g(0) = 0$ and $g(x+i)_r = (i+1)_r$ for all $r = 1, 2$. Then we can prove that $fgf = f$. Therefore, for odd number n, $n \geq 3$, $W_n(2)$ is endo-regular. □

From Lemma 2.6, Lemma 2.7 and Lemma 2.11, then

Theorem 2.12. *A generalized wheel graph $W_n(m)$ is endo-regular if and only if n is odd and $m = 2$.*

3 Endo-Orthodox and Endo-Completely-Regularity of Generalized Wheel Graphs

This section, we characterize the endo-orthodox and endo-completely-regularity of generalized wheel graphs $W_n(2)$, where n is odd, $n \geq 3$.

Lemma 3.1. *Let n be odd and $f \in End(W_n(2))$. Then f is idempotent if and only if $f|_{W_n(1)}$ is identity map.*

Proof. Necessity. Let $f \in End(W_n(2))$ be an idempotent. From Lemma 2.3, $f(0) \in V(G_1) \cup \{0\}$. Suppose $f(0) = x_1$ for some $x_1 \in V(G_1)$. Then $f(x_1) = f^2(0) = f(0) = x_1$, which is impossible because $\{0, x_1\} \in E(W_n(m))$ but $\{f(0), f(x_1)\} = \{x_1, x_1\} \notin E(W_n(2))$. Therefore, $f(0) = 0$ and $f(x_1) \in V(G_1)$ for all $x_1 \in V(G_1)$. Since $End(G_1) \cong End(C_n)$, f is one to one. So f must be in the form $f(x_1) = (x+k)_1$ for all $x = 1, \ldots, n$ for some $k = 0, 1, \ldots, n-1$. Then $(x+k)_1 = f(x_1) = f^2(x_1) = f((x+k)_1) = (x+k+k)_1$. Thus $k = 0$. Therefore, $f(x_1) = x_1$, i.e. $f|_{W_n(1)}$ is the identity map.

Sufficiency. Let $f \in End(W_n(2))$ and $f|_{W_n(1)}$ be the identity map. Then $f(x_2) \in \{0, (x-1)_1, (x+1)_1, x_2\}$. Thus $f^2(x_2) = f(x_2)$. Therefore, f is idempotent. □

Theorem 3.2. *A generalized wheel graph $W_n(m)$ is endo-orthodox if and only if n is odd and $m = 2$.*

Proof. Necessity. Let $W_n(m)$ is endo-orthodox. Then by Theorem 2.12, n is odd and $m = 2$.

Sufficiency. Consider $W_n(2)$ where n is odd. Let $f, g \in End(W_n(2))$ be idempotents. Then by Lemma 3.1, $f|_{W_n(1)}$ and $g|_{W_n(1)}$ are identity maps. Thus $fg|_{W_n(1)}$ is also identity map. Again by Lemma 3.1, fg is idempotent. Therefore, $W_n(2)$ is endo-orthodox. □

The next part we will show that only the generalized wheel graph $W_3(2)$ is endo-completely-regular .

Lemma 3.3. *The generalized wheel graph $W_n(2)$ is not endo-completely-regular for all odd integer n, $n > 3$.*

Proof. Let n be an odd integer, $n > 3$. Define $f \in End(W_n(2))$ by $f(0) = f(n_2) = 0$ and for each $x_i \in V(W_n(2)) \setminus \{0, n_2\}$,

$$f(x_i) = \begin{cases} (x+1)_1, & i = 1; \\ 3_1, & x_i = 1_2; \\ 3_2, & x_i = 2_2; \\ x_1, & x_i \in \{3_2, \ldots, (n-1)_2\}. \end{cases}$$

Suppose there exists commuting pseudo inverse g of f. Consider $fg(3_2) = fgf(2_2) = f(2_2) = 3_2$, then $gf(3_2) = 3_2$. Therefore, $g(3_1) = 3_2$. But $fg(2_1) = fgf(1_1) = f(1_1) = 2_1$, then $gf(2_1) = 2_1$. Therefore, $g(3_1) = 2_1$. It is impossible. □

For the generalized wheel graph $W_3(2)$, let us denote the class of all non-injective endomorphisms of $W_3(2)$ by $End'(W_3(2))$. Since $Aut(W_3(2))$ forms a group and $End(W_3(2)) = End'(W_3(2)) \cup Aut(W_3(2))$, $End(W_3(2))$ is completely regular if and only if $End'(W_3(2))$ is also completely regular. Let f be an idempotent of $End'(W_3(2))$. By Lemma 2.8, $f(x_2) \in V(W_3(1))$ for all $x_2 \in V(G_2)$. By Lemma 3.1, $f|_{W_3(1)}$ is the identity map. Therefore, $f(x_2) \in V(W_3(1)) \setminus \{x_1\}$ for all $x_2 \in V(G_2)$ and the class of all non-injective idempotent endomorphisms in $W_3(2)$ is $Idpt(End'(W_3(2))) =$

$$\Big\{e_1 = \begin{pmatrix} 0 & 1_1 & 2_1 & 3_1 & 1_2 & 2_2 & 3_2 \\ 0 & 1_1 & 2_1 & 3_1 & 0 & 1_1 & 2_1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1_1 & 2_1 & 3_1 & 1_2 & 2_2 & 3_2 \\ 0 & 1_1 & 2_1 & 3_1 & 0 & 3_1 & 1_1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1_1 & 2_1 & 3_1 & 1_2 & 2_2 & 3_2 \\ 0 & 1_1 & 2_1 & 3_1 & 0 & 3_1 & 2_1 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 1_1 & 2_1 & 3_1 & 1_2 & 2_2 & 3_2 \\ 0 & 1_1 & 2_1 & 3_1 & 2_1 & 0 & 1_1 \end{pmatrix}, e_5 = \begin{pmatrix} 0 & 1_1 & 2_1 & 3_1 & 1_2 & 2_2 & 3_2 \\ 0 & 1_1 & 2_1 & 3_1 & 2_1 & 1 & 0 \end{pmatrix}, e_6 = \begin{pmatrix} 0 & 1_1 & 2_1 & 3_1 & 1_2 & 2_2 & 3_2 \\ 0 & 1_1 & 2_1 & 3_1 & 2_1 & 3_1 & 0 \end{pmatrix},$$

$$e_7 = \begin{pmatrix} 0 & 1_1 & 2_1 & 3_1 & 1_2 & 2_2 & 3_2 \\ 0 & 1_1 & 2_1 & 3_1 & 2_1 & 3_1 & 1_1 \end{pmatrix}, e_8 = \begin{pmatrix} 0 & 1_1 & 2_1 & 3_1 & 1_2 & 2_2 & 3_2 \\ 0 & 1_1 & 2_1 & 3_1 & 3_1 & 0 & 1_1 \end{pmatrix}, e_9 = \begin{pmatrix} 0 & 1_1 & 2_1 & 3_1 & 1_2 & 2_2 & 3_2 \\ 0 & 1_1 & 2_1 & 3_1 & 3_1 & 0 & 2_1 \end{pmatrix},$$

$$e_{10} = \begin{pmatrix} 0 & 1_1 & 2_1 & 3_1 & 1_2 & 2_2 & 3_2 \\ 0 & 1_1 & 2_1 & 3_1 & 3_1 & 1_1 & 0 \end{pmatrix}, e_{11} = \begin{pmatrix} 0 & 1_1 & 2_1 & 3_1 & 1_2 & 2_2 & 3_2 \\ 0 & 1_1 & 2_1 & 3_1 & 3_1 & 1_1 & 2_1 \end{pmatrix} \}.$$

Then

Lemma 3.4. *Let* $f, g \in End'(W_3(2))$. *The following statements are true:*

1. $f\mathcal{H}g \Leftrightarrow \rho_f = \rho_g$.
2. $f \in H_{e_k}$ *for some* $1 \leq k \leq 11$.
3. $f\mathcal{H}gf$.

Proof. Let $f, g \in End'(W_3(2))$.

1. Since $f, g \in End'(W_3(2))$, by Lemma 2.8, $f(x_2), g(x_2) \in V(W_3(1))$ for all $x_2 \in V(G_2)$, and $I_f = I_g$. Therefore, by Lemma 1.7, $f\mathcal{H}g \Leftrightarrow \rho_f = \rho_g$.

2. Since $f \in End'(W_3(2))$, by Lemma 2.8, $f(x_2) \in V(W_3(1))$ for all $x_2 \in V(G_2)$. Let $f(x_2) = f(y_1) \in V(W_3(1))$ where $x \neq y$, $x \in \{1,2,3\}$, $y \in \{1,2,3\} \cup \{0\}$ and $f|_{W_3(1)}$ is bijective on $W_3(1)$. Let us define a mapping e on $V(W_3(2))$ by

$$e(x) = \begin{cases} x, & \text{if } x \in V(W_3(1)); \\ y, & \text{if } x \in V(G_2) \text{ and } f(x) = f(y). \end{cases}$$

We will show that $e \in End'(W_3(2))$ and $f \in H_e$. Let $\{x_2, x_2'\} \in E(W_3(2))$. Then $e(x_2) = y_1$ and $e(x_2') = y_1'$ where $f(x_2) = f(y_1)$ and $f(x_2') = f(y_1')$. Thus $\{f(x_2), f(x_2')\} = \{f(y_1), f(y_1')\} \in E(W_3(2))$. Then $y_1 \neq y_2$, therefore, $\{y_1, y_1'\} \in E(W_3(2))$, i.e. $e \in End(W_3(2))$. Since $e|_{W_3(1)}$ is the identity map, by Lemma 3.1, e is an idempotent of $End'(W_3(2))$. Next, we show that $f \in H_e$. Since $W_3(1)$ is isomorphic to K_4, for all $x, y \in V(W_3(1))$, $x = y$ if and only if $f(x) = f(y)$. Let $x, y \in \{0,1,2,3\}$. Then $f(x_1) = f(y_1) \Leftrightarrow x_1 = y_1 \Leftrightarrow e(x_1) = e(y_1)$, $f(x_2) = f(y_2) \Leftrightarrow f(x_1') = f(y_1')$ for some $x_1', y_1' \in V(W_3(1)) \Leftrightarrow x_1' = y_1'$, i.e. $f(x_2) = f(y_2) \Leftrightarrow e(x_2) = e(y_2)$, and

$$\begin{aligned} f(x_2) = f(y_1) &\Rightarrow e(x_2) = y_1 \\ &\Rightarrow e(x_2) = e(y_1), \end{aligned}$$

and

$$\begin{aligned} f(x_2) \neq f(y_1) &\Rightarrow f(x_2) = f(y_1'),\ for\ some\ y_1' \neq y_1 \\ &\Rightarrow e(x_2) = y_1' \neq y_1 = e(y_1). \end{aligned}$$

Therefore, $\rho_f = \rho_e$, i.e., $f \in H_e$.

3. If $x_i, x'_j \in V(W_3(2))$ such that $f(x_i) = f(x'_j)$, then $gf(x_i) = gf(x'_j)$. Suppose there exist $y_i, y'_j \in V(W_3(2))$ such that $f(y_i) \neq f(y'_j)$. Let $f(y_i) = u$ and $f(y'_j) = v$ for some $u \neq v$ and $u, v \in \{0, 1_1, 2_1, 3_1\}$. From $g|_{W_3(1)}$ is bijective on $W_3(1)$. Thus $g(u) \neq g(v)$, i.e. $gf(y_i) \neq gf(y'_j)$. Therefore, $\rho_f = \rho_{gf}$. Then by 1, $fHgf$. □

From Lemma 3.4(2), it is show that the semigroup $End'(W_3(2))$ is a union of (disjoint) groups. This implies that $End'(W_3(2))$ is completely regular.

Theorem 3.5. *A generalized wheel graph $W_n(m)$ is endo-completely-regular if and only if $n = 3$ and $m = 2$.*

Moreover, by Lemma 3.4, we can show that $End'(W_3(2))$ forms a right group.

Lemma 3.6. *Let $f, f' \in H_{e_k}$ for some $1 \leq k \leq 11$. If $f|_{W_3(1)} = f'|_{W_3(1)}$, then $f = f'$.*

Proof. Let $f, f' \in H_{e_k}$ and $f|_{W_3(1)} = f'|_{W_3(1)}$, for some $k = 1, \ldots, 11$. Suppose $x_2 \in V(G_2)$ such that $f(x_2) = f(y_1)$, for some $y_1 \in V(W_3(1))$. Then $f(x_2) = f(y_1) \Leftrightarrow e_k(x_2) = e_k(y_1) \Leftrightarrow f'(x_2) = f'(y_1)$. Therefore, $f = f'$. □

Theorem 3.7. *$End'(W_3(2))$ forms a right group isomorphic to $S_4 \times R_{11}$.*

Proof. From Lemma 3.6, it is clearly that $\alpha_k : H_{e_k} \to End(W_3(1))$ defined by $\alpha(f) = f|_{W_3(1)}$, is an isomorphism. Therefore, $End(W_3(1)) \cong H_{e_k}$ for all $k = 1, \ldots, 11$. Let $\varphi : End'(W_3(2)) \to (End(W_3(1)) \times R_{11})$ be defined by $\varphi(f) = (f|_{W_3(1)}, r_k)$ where $f \in H_{e_k}$. Let $f \in H_{e_k}$ and $g \in H_{e_l}$. By Lemma 3.4(3), $gf \in H_{e_k}$. Then $\varphi(gf) = (gf|_{W_3(1)}, r_k) = (g|_{W_3(1)}, r_l)(f|_{W_3(1)}, r_k) = \varphi(g)\varphi(f)$. Therefore φ is a homomorphism and from Lemma 3.6, φ is also one to one and onto. Therefore, $End'(W_3(2)) \cong (End(W_3(1)) \times R_{11})$. Since $End(W_3(1))$ is isomorphic to the group S_4, hence $End'(W_3(2)) \cong S_4 \times R_{11}$. □

Corollary 3.8. *$|End'(W_3(2))| = 11 \cdot 4! = 264$ and $|End(W_3(2))| = 270$.*

Remark 3.9. For each non-injective $f \in End'(W_3(2))$, let $f_1 = f|_{W_3(1)}$. Then $g : V(W_3(2)) \to V(W_3(2))$ which is defined by $g(0) = f_1(0)$, and

$$g(x_i) = \begin{cases} f_1^{-1}(x_i), & \text{if } i = 1; \\ f_1^{-1}f_1^{-1}(f(x_i)), & \text{if } i = 2, \end{cases}$$

is a commuting pseudo inverse.

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