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Module Amenability and Weak Module Amenability for Second Dual of Banach Algebras

Massoud Amini and Abasalt Bodaghi*

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Abstract: In this paper we define the weak module amenability of a Banach algebra $\mathcal{A}$ which is a Banach module over another Banach algebra $\mathfrak{A}$ with compatible actions, and show that under some mild conditions weak module amenability of $\mathcal{A}^{**}$ implies weak module amenability of $\mathcal{A}$. Also among other results, we investigate the relation between module Arens regularity of a Banach algebra and module amenability of its second dual. As a consequence we prove that $\ell^1(S)$ is always weakly module amenable (as an $\ell^1(E)$-module), where S is an inverse semigroup with an upward directed set of idempotents E.

Keywords: Banach modules, weak amenability, module amenability, weak module amenability, semigroup algebra, inverse semigroup

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1 Introduction

A Banach algebra $\mathcal{A}$ is *amenable* if every bounded derivation from $\mathcal{A}$ into any dual Banach A-module is inner, equivalently if $H^1(\mathcal{A}, X^*) = \{0\}$ for every Banach A-module X, where $H^1(\mathcal{A}, X^*)$ is the *first Hochschild cohomology group* of A with

* *Corresponding author*

coefficients in X^*. This concept was introduced by Barry Johnson in [20]. The notion of weak amenability was introduced by W. G. Bade, P. C. Curtis and H. G. Dales in [4] for commutative Banach algebras. Later Johnson defined weak amenability for arbitrary Banach algebras [22]. In fact a Banach algebra $\mathcal{A}$ is *weakly amenable* if $H^1(\mathcal{A},\mathcal{A}^*)=\{0\}$.

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{A}^{**}$ be the second dual of $\mathcal{A}$. It is known that the Banach algebra $\mathcal{A}$ inherits amenability from $\mathcal{A}^{**}$ [18] (see also [17]). The first author in [1] introduced the concept of module amenability for Banach algebras which are Banach modules on another Banach algebra with compatible actions. This could be considered as a generalization of the Johnson's amenability. It is shown in [2] that if $\mathcal{A}$ is a commutative Banach $\mathfrak{A}$-module such that $\mathcal{A}^{**}$ is $\mathfrak{A}$-module amenable, then $\mathcal{A}$ is module amenable. The authors improved this result for the concept of module biflatness in [5], by assuming a weaker condition on $\mathcal{A}$.

The analogous result for weak amenability is not known in general, but it is known to hold for Banach algebras $\mathcal{A}$ which are left ideal in $\mathcal{A}^{**}$ [17] (see also [7]), the dual Banach algebras [16], the Banach algebras $\mathcal{A}$ which are Arens regular and every derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is weakly compact [13], Banach algebras for which the second adjoint of each derivation $D:\mathcal{A}\to\mathcal{A}^*$ satisfies $D''(\mathcal{A}^{**})\subseteq \mathrm{WAP}(\mathcal{A})$, and Banach algebras $\mathcal{A}$ which are right ideals in $\mathcal{A}^{**}$ and satisfy $\mathcal{A}^{**}\mathcal{A}=\mathcal{A}^{**}$ [15]. For the latter case, using the concept of module structures on iterated duals, an alternative proof is given in [6].

A discrete semigroup S is called *amenable* if there exists a mean m on $\ell^\infty(S)$ which is both left and right invariant (see [14]). An *inverse semigroup* is a discrete semigroup S such that for each $s\in S$, there is a unique element $s^*\in S$ with $ss^*s=s$ and $s^*ss^*=s^*$. Elements of the form ss^* are called *idempotents* of S. For an inverse semigroup S, a left invariant mean on $\ell^\infty(S)$ is right invariant and vise versa.

Amini and Ebrahimi Bagha in [3] defined the concept of weak module amenability for a Banach algebra which is a commutative Banach module over another Banach algebra and showed that if S is commutative semigroup with the set of idempotents E, under a natural action, $\ell^1(S)$ is always weak $\ell^1(E)$-module amenable. In this paper we define weak module amenability in a more general context, for a Banach algebra which is not necessarily a commutative Banach module, and show that if S is an inverse semigroup with an upward directed set of idempotents E, then $\ell^1(S)$ as an $\ell^1(E)$-module is weak module amenable. This could be considered as the module version (for inverse semigroups) of a result of Johnson [21] which asserts that for any locally compact group G, the group algebra $L^1(G)$ is weakly amenable.

We also prove that when $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from left, then under some mild conditions, weak module amenability $\mathcal{A}^{**}$ implies weak amenability $\mathcal{A}^{**}/J^{\perp\perp}$, where J is the closed ideal of $\mathcal{A}$ generated by $\alpha\cdot(ab)-(ab)\cdot\alpha$ for all $a\in\mathcal{A}$ and $\alpha\in\mathfrak{A}$. As a consequence we show that under some conditions on the Banach algebra $\mathcal{A}/J$, weak module amenability of $\mathcal{A}^{**}$ implies weak module amenability of $\mathcal{A}$. Finally, we find a relation between module Arens regularity of a Banach algebra $\mathcal{A}$ with module topological centers and module amenability of $\mathcal{A}^{**}$.

2 Module Amenability

Throughout this paper, $\mathcal{A}$ and $\mathfrak{A}$ are Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha\cdot(ab)=(\alpha\cdot a)b,\ (ab)\cdot\alpha=a(b\cdot\alpha)\quad(a,b\in\mathcal{A},\alpha\in\mathfrak{A}).$$

Let X be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha\cdot(a\cdot x)=(\alpha\cdot a)\cdot x,\ a\cdot(\alpha\cdot x)=(a\cdot\alpha)\cdot x,\ (\alpha\cdot x)\cdot a=\alpha\cdot(x\cdot a)\quad(a\in\mathcal{A},\alpha\in\mathfrak{A},x\in X)$$

and the same for the right or two-sided actions. Then we say that X is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. If moreover

$$\alpha\cdot x=x\cdot\alpha\quad(\alpha\in\mathfrak{A},x\in X)$$

then X is called a *commutative* $\mathcal{A}$-$\mathfrak{A}$-module. If X is a (commutative) Banach $\mathcal{A}$-$\mathfrak{A}$-module, then so is X^*, where the actions of $\mathcal{A}$ and $\mathfrak{A}$ on X^* are defined by

$$\langle\alpha\cdot f,x\rangle=\langle f,x\cdot\alpha\rangle,\ \langle a\cdot f,x\rangle=\langle f,x\cdot a\rangle\quad(a\in\mathcal{A},\alpha\in\mathfrak{A},x\in X,f\in X^*)$$

and the same for the right actions. Let Y be another $\mathcal{A}$-$\mathfrak{A}$-module, then a $\mathcal{A}$-$\mathfrak{A}$-module morphism from X to Y is a norm-continuous map $\varphi:X\longrightarrow Y$ with $\varphi(x\pm y)=\varphi(x)\pm\varphi(y)$ and

$$\varphi(\alpha\cdot x)=\alpha\cdot\varphi(x),\ \varphi(x\cdot\alpha)=\varphi(x)\cdot\alpha,\ \varphi(a\cdot x)=a\cdot\varphi(x),\varphi(x\cdot a)=\varphi(x)\cdot a,$$

for $x,y\in X,a\in\mathcal{A},$ and $\alpha\in\mathfrak{A}$.

Note that when $\mathcal{A}$ acts on itself by algebra multiplication, it is not in general a Banach $\mathcal{A}$-$\mathfrak{A}$-module, as we have not assumed the compatibility condition

$$a \cdot (\alpha \cdot b) = (a \cdot \alpha) \cdot b \quad (\alpha \in \mathfrak{A}, a, b \in \mathcal{A}).$$

If $\mathcal{A}$ is a commutative $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then it is also a Banach $\mathcal{A}$-$\mathfrak{A}$-module.

Consider the projective tensor product $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$. It is well known that $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$ is a Banach algebra with respect to the canonical multiplication map defined by

$$(a \otimes b)(c \otimes d) = (ac \otimes bd)$$

and extended by bi-linearity and continuity [9]. Then $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module with canonical actions. Let I be the closed ideal of the projective tensor product $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$ generated by elements of the form $\alpha \cdot a \otimes b - a \otimes b \cdot \alpha$ for $\alpha \in \mathfrak{A}, a, b \in \mathcal{A}$. Consider the map $\omega \in \mathcal{L}(\mathcal{A}\widehat{\bigotimes}\mathcal{A}, \mathcal{A})$ defined by $\omega(a \otimes b) = ab$ and extended by linearity and continuity. Let J be the closed ideal of $\mathcal{A}$ generated by $\omega(I)$. Then the module projective tensor product $\mathcal{A}\widehat{\bigotimes}_{\mathfrak{A}}\mathcal{A} \cong (\mathcal{A}\widehat{\bigotimes}\mathcal{A})/I$ and the quotient Banach algebra $\mathcal{A}/J$ are Banach $\mathfrak{A}$-modules with compatible actions. We have $(\mathcal{A}\widehat{\bigotimes}_{\mathfrak{A}}\mathcal{A})^* = \mathcal{L}_{\mathfrak{A}}(\mathcal{A}, \mathcal{A}^*)$ where the right hand side is the space of all $\mathfrak{A}$-module morphism from A to A^* [29]. Also the map $\widetilde{\omega} \in \mathcal{L}(\mathcal{A}\widehat{\bigotimes}_{\mathfrak{A}}\mathcal{A}, \mathcal{A}/J)$ defined by $\widetilde{\omega}(a \otimes b + I) = ab + J$ extends to an $\mathfrak{A}$-module morphism.

Let $\mathcal{A}$ and $\mathfrak{A}$ be as in the above and X be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. Let I and J be the corresponding closed ideals of $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$ and $\mathcal{A}$, respectively. A bounded map $D : \mathcal{A} \longrightarrow X$ is called a *module derivation* if

$$D(a \pm b) = D(a) \pm D(b), \quad D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathcal{A}),$$

and

$$D(\alpha \cdot a) = \alpha \cdot D(a), \quad D(a \cdot \alpha) = D(a) \cdot \alpha \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

Although D is not necessary linear, but still its boundedness implies its norm continuity (since it preserves subtraction). When X is commutative $\mathcal{A}$-$\mathfrak{A}$-module, each $x \in X$ defines a module derivation

$$D_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}).$$

These are called *inner* module derivations. The Banach algebra $\mathcal{A}$ is called *module amenable* (as an $\mathfrak{A}$-module) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X, each module derivation $D : \mathcal{A} \longrightarrow X^*$ is inner [1].

Let $\Box$ and $\Diamond$ be the first and second Arens products on the second dual space $\mathcal{A}^{**}$, then $\mathcal{A}^{**}$ is a Banach algebra with respect to both of these products. When these two products coincide on $\mathcal{A}^{**}$, we say that $\mathcal{A}$ is *Arens regular*, and when they coincide only on $\mathcal{A}$ we say that $\mathcal{A}$ is *strongly Arens irregular*. Closed subalgebras of the algebra $\mathcal{B}(H)$ of bounded operators on a Hilbert space H are Arens regular, whereas the group algebra $L^1(G)$ of a locally compact group G is strongly Arens irregular [11, 23].

The *first module topological center* of $\mathcal{A}^{**}$ (as an $\mathfrak{A}$-module) is

$$\mathcal{Z}_{\mathfrak{A}}^{(1)}(\mathcal{A}^{**}) = \{G \in \mathcal{A}^{**} : F \longrightarrow G\Box F \quad \text{is} \ \ \sigma(\mathcal{A}^{**}, J^{\perp})\text{-continuous}\}.$$

It is shown in [2] that

$$\mathcal{Z}_{\mathfrak{A}}^{(1)}(\mathcal{A}^{**}) = \{G \in \mathcal{A}^{**} : G\Box F - G\Diamond F \in J^{\perp\perp} \ \ \forall F \in \mathcal{A}^{**}\}.$$

Also, like in the classic case, we can define the *second module topological center* of $\mathcal{A}^{**}$ by

$$\mathcal{Z}_{\mathfrak{A}}^{(2)}(\mathcal{A}^{**}) = \{G \in \mathcal{A}^{**} : F \longrightarrow F\Diamond G \quad \text{is} \ \ \sigma(\mathcal{A}^{**}, J^{\perp})\text{-continuous}\}.$$

It is shown in [2, Proposition 2.2], $\mathcal{A}$ is module Arens regular if and only if $\mathcal{Z}_{\mathfrak{A}}^{(1)}(\mathcal{A}^{**}) = \mathcal{A}^{**}$, or equivalently $\mathcal{Z}_{\mathfrak{A}}^{(2)}(\mathcal{A}^{**}) = \mathcal{A}^{**}$. Also $\mathcal{Z}_{\mathfrak{A}}^{(1)}(\mathcal{A}^{**})$ and $\mathcal{Z}_{\mathfrak{A}}^{(1)}(\mathcal{A}^{**})$ are $\sigma(\mathcal{A}^{**}, J^{\perp})$-closed subalgebras of $(\mathcal{A}^{**}, \Box)$ containing $\mathcal{A}$. Now suppose that $\mathcal{A}$ is a commutative Banach $\mathfrak{A}$-module, then $J^{\perp\perp} = 0$. Hence for each α in $\mathfrak{A}$ and F, G in $\mathbf{B}$, we have $f\Box G = G\Diamond F$ for all $F \in \mathcal{A}^{**}$. Thus

$$(\alpha \cdot G)\Box F = \alpha \cdot (G\Diamond F) = \alpha \cdot (G\Diamond F) = (\alpha \cdot G)\Diamond F) \quad (F \in \mathcal{A}^{**}).$$

Therefore $\mathcal{Z}_{\mathfrak{A}}^{(1)}(\mathcal{A}^{**})$ is an $\mathfrak{A}$-submodule of $\mathcal{A}^{**}$. Similarly, for $\mathcal{Z}_{\mathfrak{A}}^{(2)}(\mathcal{A}^{**})$.

Definition 2.1. [1] *A bounded net* $\{\widetilde{\xi_j}\}$ *in* $\mathcal{A}\widehat{\bigotimes}_{\mathfrak{A}}\mathcal{A}$ *is called a module approximate diagonal if* $\widetilde{\omega}_{\mathcal{A}}(\widetilde{\xi_j})$ *is a bounded approximate identity of* $\mathcal{A}/J$ *and*

$$\lim_j \|\xi_j \cdot a - a \cdot \xi_j\| = 0 \quad (a \in \mathcal{A}).$$

An element $\widetilde{E} \in (\mathcal{A}\widehat{\bigotimes}_{\mathfrak{A}}\mathcal{A})^{**}$ *is called a module virtual diagonal if*

$$\widetilde{\omega}_{\mathcal{A}}^{**}(\widetilde{E}) \cdot a = \widetilde{a}, \quad \widetilde{E} \cdot a = a \cdot \widetilde{E} \quad (a \in \mathcal{A}),$$

where $\widetilde{a} = a + J^{\perp\perp}$.

Assume that $\mathbf{B}$ is a closed subalgebra and an $\mathfrak{A}$-submodule of $\mathcal{A}^{**}$ such that $\widehat{\mathcal{A}} \subseteq \mathbf{B}$. Consider the module projective tensor product $\mathbf{B}\widehat{\bigotimes}_{\mathfrak{A}}\mathbf{B}$, that is $\mathbf{B}\widehat{\bigotimes}_{\mathfrak{A}}\mathbf{B} = (\mathbf{B}\widehat{\bigotimes}\mathbf{B})/M_{\mathbf{B}}$, where $M_{\mathbf{B}}$ is a closed ideal generated by elements of the form $\alpha \cdot F \otimes G - F \otimes G \cdot \alpha$, for $\alpha \in \mathfrak{A}, F, G \in \mathbf{B}$. We denote $M_{\mathcal{A}^{**}}$ by M. Also for each α in $\mathfrak{A}$ and F, G in $\mathbf{B}$, we consider $N_{\mathbf{B}}$ to be the closed ideal of $\mathbf{B}$ generated by $(\alpha \cdot F)\Box G - F\Box(G \cdot \alpha)$. We denote $N_{\mathcal{A}^{**}}$ by N. It is shown in the proof of [2, Theorem 3.4] that the map $\lambda : \mathcal{A}^{**}/N \longrightarrow \mathcal{A}^{**}/J^{\perp\perp}; F + N \mapsto F + J^{\perp\perp}$ is a surjective bounded $\mathcal{A}$-$\mathfrak{A}$-module morphism.

Theorem 2.2. *Let $\mathcal{A}/J$ be a commutative Banach $\mathfrak{A}$-module and $\mathbf{B}/N_{\mathbf{B}}$ be a commutative $\mathfrak{A}$-module (or $N_{\mathbf{B}}$ is w^*-closed) such that $\widehat{\mathcal{A}} \subseteq \mathbf{B}$, then module amenability $\mathbf{B}$ implies module amenability $\mathcal{A}$.*

Proof. It is shown in [2] that there is a continuous linear mapping

$$\Phi_{\mathfrak{A}} : \mathcal{A}^{**}\widehat{\bigotimes}\mathcal{A}^{**}/M \longrightarrow (\mathcal{A}\widehat{\bigotimes}\mathcal{A})^{**}/I^{\perp\perp}$$

such that for $a, b, x \in \mathcal{A}$ and $m \in \mathcal{A}^{**}\widehat{\bigotimes}\mathcal{A}^{**}$ the following equalities hold.

(1) $\Phi_{\mathfrak{A}}(a \otimes b + M) = a \otimes b + I^{\perp\perp}$,

(2) $\Phi_{\mathfrak{A}}(m + M) \cdot x = \Phi_{\mathfrak{A}}(m \cdot x + M)$,

(3) $x \cdot \Phi_{\mathfrak{A}}(m + M) = \Phi_{\mathfrak{A}}(x \cdot m + M)$,

(4) $\tilde{\omega}_{\mathcal{A}}^{**}(\Phi_{\mathfrak{A}}(m + M)) = \lambda \circ \tilde{\omega}_{\mathcal{A}^{**}}(m + M)$.

Consider the linear map $\phi : \mathcal{A}/J \longrightarrow \mathbf{B}/N_{\mathbf{B}}; (a + J \mapsto a + N_{\mathbf{B}})$. It is known that every norm-norm-continuous map is weak*-weak*-continuous. Hence ϕ is weak*-weak*-continuous. We show that $\phi(\mathcal{A}/J)$ is weak*-dense in $\mathbf{B}/N_{\mathbf{B}}$. Let $F \in \mathbf{B}$ and bounded net (a_j) be in $\mathcal{A}$ such that $\widehat{a}_j \stackrel{w^*}{\longrightarrow} F$. Since $N_{\mathbf{B}} \subseteq \mathbf{B} \subseteq \mathcal{A}^{**}$, $N_{\mathbf{B}}$ has the annihilator ${}^{\perp}N_{\mathbf{B}} = \{f \in \mathcal{A}^* : \langle G, f\rangle = 0 \text{ for all } G \in N_{\mathbf{B}}\}$. We have

$$({}^{\perp}N_{\mathbf{B}})^* = \mathbf{B}/({}^{\perp}N_{\mathbf{B}})^{\perp} = \mathbf{B}/\overline{N_{\mathbf{B}}}^{w^*} = \mathbf{B}^{**}/N_{\mathbf{B}}.$$

Hence $\mathbf{B}^{**}/N_{\mathbf{B}}$ is a dual algebra. Suppose that f is in ${}^{\perp}N_{\mathbf{B}}$, then $f(a_j) \longrightarrow F(f)$. Since $f\,|_{N_{\mathbf{B}}} = 0$, f lifts to a map, still denoted by f, on $\mathbf{B}^{**}/N_{\mathbf{B}}$. Thus we deduce that $\langle a_j + N_{\mathbf{B}}, f\rangle \longrightarrow \langle F + N_{\mathbf{B}}, f\rangle$. Since $\mathcal{A}/J$ is commutative $\mathfrak{A}$-module, $\mathbf{B}/N_{\mathbf{B}}$ is also a commutative $\mathfrak{A}$-module. Also since $\mathbf{B}$ is module amenable, by [5, Corrolary 2.11] $\mathbf{B}/N_{\mathbf{B}}$ has a bounded approximate identity. Hence $\mathbf{B}$ has a module approximate diagonal $\{\widetilde{m_i}\}$ ($\widetilde{m_i} = m_i + M_{\mathbf{B}}$) in $\mathbf{B}\widehat{\bigotimes}_{\mathfrak{A}}\mathbf{B}$ by [1, Theorem 2.1], and so $\tilde{\omega}_{\mathbf{B}}(\widetilde{m_i})\overline{b} \longrightarrow \overline{b}$ and $\widetilde{m_i} \cdot b - b \cdot \widetilde{m_i} \longrightarrow 0$, where $\overline{b} = b + N_{\mathbf{B}}$ whenever

$b \in \mathbf{B}$. Consider the map

$$T : \mathbf{B}\widehat{\bigotimes}\mathbf{B}/M_{\mathbf{B}} \longrightarrow \mathcal{A}^{**}\widehat{\bigotimes}\mathcal{A}^{**}/M; \quad b_1 \otimes b_2 + M_{\mathbf{B}} \mapsto b_1 \otimes b_2 + M \quad (b_1, b_2 \in \mathbf{B}).$$

Obviously, this map is well-defined and norm decreasing. Set $\Theta = \Phi_{\mathfrak{A}} \circ T$. Now consider the map $\theta : \mathbf{B}/N_{\mathbf{B}} \longrightarrow \mathcal{A}^{**}/N$ defined by $\theta(F + N_{\mathbf{B}}) = F + N$; $(F \in \mathbf{B})$. The map θ is well-defined and continuous homomorphism. From the above equalities, for each $a \in \mathcal{A}$, we have

$$\begin{aligned}
\tilde{\omega}_{\mathcal{A}}^{**}(\Theta(\widetilde{m_i})) \cdot \widetilde{a} = \tilde{\omega}_{\mathcal{A}}^{**}(\Phi_{\mathfrak{A}}(T(\widetilde{m_i}))) \cdot \widetilde{a} &= \lambda \circ \tilde{\omega}_{\mathcal{A}^{**}}(m_i + M) \cdot \widetilde{a} \\
&= \lambda(\theta(\tilde{\omega}_{\mathbf{B}}(\widetilde{m_i}))) \cdot \widetilde{a} = \lambda((\theta(\tilde{\omega}_{\mathbf{B}}(\widetilde{m_i}))) \cdot (a + N)) \\
&= \lambda(\theta(\tilde{\omega}_{\mathbf{B}}(\widetilde{m_i}) \cdot (a + N_{\mathbf{B}}))) \\
&\longrightarrow \lambda(\theta(a + N_{\mathbf{B}})) = \lambda(a + N) = \widetilde{a},
\end{aligned}$$

and

$$\begin{aligned}
a \cdot \Theta(\widetilde{m_i}) - \Theta(\widetilde{m_i}) \cdot a &= a \cdot \Phi_{\mathfrak{A}} \circ T(\widetilde{m_i}) - \Phi_{\mathfrak{A}} \circ T(\widetilde{m_i}) \cdot a \\
&= a \cdot \Phi_{\mathfrak{A}}(m_i + M) - \Phi_{\mathfrak{A}}(m_i + M) \cdot a \\
&= \Phi_{\mathfrak{A}}(a \cdot m_i - m_i \cdot a + M) \\
&= \Phi_{\mathfrak{A}} \circ T(a \cdot \widetilde{m_i} - \widetilde{m_i} \cdot a) \\
&\longrightarrow 0.
\end{aligned}$$

Assume that the bounded net $(\Theta(\widetilde{m_i})) \subseteq (\mathcal{A}\widehat{\bigotimes}_{\mathfrak{A}}\mathcal{A})^{**}$ has the w^*-cluster point $\widetilde{E}$. Hence for each $a \in \mathcal{A}$ we have

$$\widetilde{\omega}_{\mathcal{A}}^{**}(\widetilde{E}) \cdot a = \widetilde{a}, \quad \widetilde{E} \cdot a = a \cdot \widetilde{E}$$

Therefore $\widetilde{E}$ is a module virtual diagonal for $\mathcal{A}$. Now it follows from [1, Theorem 2.1] that $\mathcal{A}$ is module amenable. □

Corollary 2.3. *Let $\mathcal{A}$ be a commutative Banach $\mathfrak{A}$-module. If $\mathcal{Z}_{\mathfrak{A}}^{(1)}(\mathcal{A}^{**})$ or $\mathcal{Z}_{\mathfrak{A}}^{(2)}(\mathcal{A}^{**})$ is module amenable, then $\mathcal{A}$ is module amenable.*

Recall that Banach algebra $\mathcal{A}^*$ is said to *factor on the left* if $\mathcal{A}^* \cdot \mathcal{A} = \mathcal{A}^*$ [26]. Now if $\mathcal{A}$ has a bounded approximate identity and $\mathcal{A}^{**}$ has an identity, $\mathcal{A}^*$ factors on the left [26].

Proposition 2.4. *Let $\mathcal{A}/J$ be a commutative Banach $\mathfrak{A}$-module and $\mathcal{A}^{**}/N$ be commutative $\mathfrak{A}$-module (or N is w^*-closed). If $(\mathcal{A}^{**}, \Box)$ is module amenable and $\widehat{\mathcal{A}} \Box \mathcal{A}^{**} \subseteq \mathcal{Z}_{\mathfrak{A}}^{(1)}(\mathcal{A}^{**})$, then $\mathcal{A}$ is module Arens regular.*

Proof. Since $\mathcal{A}/J$ is a commutative $\mathfrak{A}$-module and $\mathcal{A}^{**}$ is module amenable, $\mathcal{A}^{**}/N$ has a bounded approximate identity [5, Corrolary 2.11]. Using the map λ, $\mathcal{A}^{**}/^{\perp\perp}$ has a bounded approximate identity, and so has an identity. Also module amenability of $\mathcal{A}^{**}$ implies module amenability of $\mathcal{A}$ [5, Theorem 3.4]. Again from [5, Corrolary 2.11], $\mathcal{A}/J$ has bounded approximate identity. Without loss of generality we may assume that $J^{\perp}$ factors on the left, that is $J^{\perp}\cdot\mathcal{A}=J^{\perp}$. Let $f\in J^{\perp}$ and $F,G\in\mathcal{A}^{**}$. Then there exists $g\in J^{\perp}$ and $a\in\mathcal{A}$ such that $f=g\cdot a$. We have

$$\begin{aligned}\langle F\Box G,f\rangle &= \langle F\Box G,g\cdot a\rangle=\langle(\widehat{a}\Box F)\Box G,g\rangle\\ &= \langle(\widehat{a}\Box F)\Diamond G,g\rangle=\langle(\widehat{a}\Diamond F)\Diamond G,g\rangle\\ &= \langle F\Diamond G,g\cdot a\rangle=\langle F\Diamond G,f\rangle.\end{aligned}$$

This complete the proof. □

Corollary 2.5. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module and $J^{\perp}\cdot\mathcal{A}=J^{\perp}$. If $\widehat{\mathcal{A}}\Box\mathcal{A}^{**}\subseteq\mathcal{Z}_{\mathfrak{A}}^{(1)}(\mathcal{A}^{**})$, then $\mathcal{A}$ is module Arens regular.*

Corollary 2.6. *Let $\mathcal{A}$ be a commutative Banach $\mathfrak{A}$-module. If $(\mathcal{A}^{**},\Box)$ is module amenable and $\widehat{\mathcal{A}}\Box\mathcal{A}^{**}\subseteq\mathcal{Z}_{\mathfrak{A}}^{(1)}(\mathcal{A}^{**})$, then $\mathcal{A}$ is module Arens regular.*

3 Weak Module Amenability

In this section, we define the concept of weak module amenability in a more general context than [3]. The previous definition [3, definition 2.2] works well only when $\mathcal{A}$ is a commutative $\mathfrak{A}$-module. Throughout this section, we assume that the Banach algebra $\mathfrak{A}$ is commutative unless otherwise stated explicitly. We refer the reader to [12, 24] for more details about the classical notion of weak amenability of Banach algebras.

Definition 3.1. *The Banach algebra $\mathcal{A}$ is called weakly module amenable (as an $\mathfrak{A}$-module) if for any commutative Banach $\mathfrak{A}$-submodule Y of $\mathcal{A}^*$, each module derivation from $\mathcal{A}$ to Y is inner.*

If $D:\mathcal{A}\longrightarrow Y\subseteq\mathcal{A}^*$ is a module derivation, then for each $\alpha\in\mathfrak{A}$ and $a,c,d\in\mathcal{A}$ we have

$$\langle D(a),\alpha\cdot cd-cd\cdot\alpha\rangle=\langle\alpha\cdot D(a)-D(a)\cdot\alpha,cd\rangle=0.$$

By continuity of D, we see $D(a) \subseteq J^{\perp}$. Hence D could be considered as a module derivation from $\mathcal{A}$ to $(\mathcal{A}/J)^* = J^{\perp}$ when $\mathcal{A}/J$ is commutative Banach $\mathfrak{A}$-module. Therefore we have the following result.

Proposition 3.2. *If $\mathcal{A}/J$ is a commutative Banach $\mathfrak{A}$-module, then the following are equivalent:*

i) $\mathcal{A}$ is weakly module amenable.

ii) Every module derivation from $\mathcal{A}$ to $(\mathcal{A}/J)^*$ is inner.

When $\mathcal{A}$ is a commutative $\mathfrak{A}$-module, then $J = \{0\}$. Hence the above definition is a generalization of [3, definition 2.2]. In fact there exist some Banach algebras which are not commutative module but a quotient of them are commutative module. The following lemma is proved in [2, Lemma 3.1].

Lemma 3.3. *Let $\mathcal{A}$ be a Banach algebra and Banach $\mathfrak{A}$-module with compatible actions, and J_0 be a closed ideal of $\mathcal{A}$ such that $J \subseteq J_0$. If $\mathcal{A}/J_0$ has a left or right identity $e + J_0$, then for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$ we have $a \cdot \alpha - \alpha \cdot a \in J_0$, i.e., $\mathcal{A}/J_0$ is commutative Banach $\mathfrak{A}$-module.*

Proposition 3.4. *Let $\mathcal{A}$ be an $\mathfrak{A}$-module, and $\mathcal{A}/J$ has an identity. If $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$, then weak amenability of $\mathcal{A}/J$ implies weak module amenability of $\mathcal{A}$.*

Proof. Suppose that $D : \mathcal{A} \longrightarrow (\mathcal{A}/J)^*$ is a module derivation. Define $\widetilde{D}$: $\mathcal{A}/J \longrightarrow (A/J)^*$ via $\widetilde{D}(a + J) = D(a)$. For each $\alpha \in \mathfrak{A}$ and $a, b \in \mathcal{A}$ we have

$$D(\alpha \cdot ab - ab \cdot \alpha) = \alpha \cdot D(ab) - D(ab) \cdot \alpha = 0.$$

On the other hand, Since J is a closed ideal, the restriction of D to J is zero. Therefore $\widetilde{D}$ is well-defined. For $a, b \in \mathcal{A}$ we have $\tilde{D}((a + J) \pm (b + J)) = \tilde{D}(a + J) \pm \tilde{D}(b + J)$ and $\tilde{D}(ab + J) = \tilde{D}(a + J) \cdot (b + J) + (a + J) \cdot \tilde{D}(b + J)$. We note that if $\mathcal{A}/J$ has identity, then it is always commutative $\mathfrak{A}$-module (see Lemma 3.3). Suppose that $e + J$ is identity $\mathcal{A}$ and $\mathfrak{A}$ has a bounded approximate identity (γ_i) for $\mathcal{A}$. for each $\lambda \in \mathbb{C}$ we have

$$\gamma_i \cdot e + J = e \cdot \gamma_i + J = \lambda e \cdot \gamma_i + J,$$

so $(\lambda e) \cdot \gamma_i - \gamma_i \cdot e \longrightarrow \lambda e - e$ in norm. Since J is a closed ideal of $\mathcal{A}$, $\lambda e - e \in J$.

For $\lambda \in \mathbb{C}, a \in \mathcal{A}$, we have

$$\begin{aligned}\tilde{D}(\lambda(a+J)) &= \tilde{D}((a+J)(\lambda e+J))\\ &= (a+J)\cdot\tilde{D}(\lambda e+J)+\tilde{D}(a+J)(\lambda e+J)\\ &= (a+J)\cdot\tilde{D}(e+J)+\lambda\tilde{D}(a+J)\cdot(e+J)\\ &= \lambda\tilde{D}(a+J).\end{aligned}$$

Thus $\tilde{D}$ is $\mathbb{C}$-linear, and so it is inner. Thus D is inner. □

Now, we prove that the semigroup algebra $\ell^1(S)$ is always $\ell^1(E)$-module weak amenable, where E is the set of idempotents of S, acting on S trivially from left and by multiplication from right. Throughout this section S is an inverse semigroup with set idempotent E, where the order of E is defined by

$$e \leq d \Longleftrightarrow ed = e \quad (e, d \in E).$$

It is easy to show that E is a (commutative) subsemigroup of S [19, Theorem V.1.2]. In particular $\ell^1(E)$ could be regarded as a subalgebra of $\ell^1(S)$, and thereby $\ell^1(S)$ is a Banach algebra and a Banach $\ell^1(E)$-module with compatible actions [1]. Here we let $\ell^1(E)$ act on $\ell^1(S)$ by multiplication from right and trivially from left, that is

$$\delta_e\cdot\delta_s=\delta_s,\ \delta_s\cdot\delta_e=\delta_{se}=\delta_s*\delta_e \quad (s\in S, e\in E).$$

We see that $\ell^1(S)$ is not commutative $\ell^1(E)$-module. In this case, the ideal J (see section 2) is the closed linear span of

$$\{\delta_{set}-\delta_{st} \quad s,t\in S, e\in E\}.$$

We consider an equivalence relation on S as follows

$$s\approx t \Longleftrightarrow \delta_s-\delta_t\in J \ \ (s,t\in S).$$

Recall that E is called *upward directed* if for every $e, f \in E$ there exist $g \in E$ such that $eg = e$ and $fg = f$. This is precisely the assertion that S satisfies the D_1 condition of Duncan and Namioka [14]. It is shown in [2] that if E is upward directed, then the quotient $S/\approx$ is a discrete group. For the semigroup algebra S, as in [27, Theorem 3.3], we may observe that $\ell^1(S)/J \cong \ell^1(S/\approx)$. Also $\ell^1(S)/J$ is a commutative $\ell^1(E)$-bimodule with the following actions

$$\delta_e\cdot(\delta_s+J)=\delta_s+J,\ (\delta_s+J)\cdot\delta_e=\delta_{se}+J \quad (s\in S, e\in E).$$

Corollary 3.5. *Let S be an inverse semigroup with an upward directed set of idempotents E. Then $\ell^1(S)$ is weakly module amenable as an $\ell^1(E)$-module with trivial left action.*

Proof. Since $S/\approx$ is a discrete group, the group algebra $\ell^1(S/\approx)$ has an identity. Also since E satisfies condition D_1 of Duncan and Namioka, $\ell^1(E)$ has a bounded approximate identity for $\ell^1(S)$ [2, 14]. Now the result follows from [21, Theorem] and Proposition 3.4 with $\mathcal{A} = \ell^1(S)$ and $\mathfrak{A} = \ell^1(E)$. □

The Banach algebras with compatible $\mathfrak{A}$-module structure could be considered as objects of a category $\mathfrak{C}_{\mathfrak{A}}$ whose morphisms are bounded $\mathfrak{A}$-module maps. We are interested in the case where $\mathfrak{A}$ is an injective object in $\mathfrak{C}_{\mathfrak{A}}$, that is for any objects $A, B \in \mathfrak{C}_{\mathfrak{A}}$ and monomorphism $\theta : B \longrightarrow A$ and morphism $\mu : B \longrightarrow \mathfrak{A}$, there exist a morphism $\widetilde{\mu} : A \longrightarrow \mathfrak{A}$ such that $\mu = \widetilde{\mu} \circ \theta$. This is the case when $\mathfrak{A} = \mathbb{C}$ (Hahn Banach Theorem). It is shown in [3] that if $\mathcal{A}$ is weakly module amenable, then span $(\mathcal{A}\mathfrak{A}\mathcal{A})$ is dense in $\mathcal{A}$. In the following proposition $\mathfrak{A}$ is not necessarily commutative, but $\mathcal{A}$ is a commutative Banach $\mathfrak{A}$-module.

Proposition 3.6. *Let $\mathfrak{A}$ be injective and has a bounded approximate identity. If $\mathcal{A}^{**}$ is weakly module amenable, then span $(\mathcal{A}\mathfrak{A}\mathcal{A})$ is dense in $\mathcal{A}$.*

Proof. Let $a \in \mathcal{A}$. Since $\mathcal{A}^{**}$ is weakly module amenable, by [3, Proposition 2.4] there exist sequences $(F_n) \subseteq (\mathcal{A}^{**}\mathfrak{A}\mathcal{A}^{**})$ and $(\alpha_n) \subseteq \mathfrak{A}$ such that $\sum_{k=1}^{p(n)}(G_{n,k} \cdot \alpha_{n,k})\Box H_{n,k}$ and norm-$\lim_n F_n = \widehat{a}$. using weak*-density of $\mathcal{A}$ in $\mathcal{A}^{**}$, there exist nets $(a_{n,k,i})$ and $(b_{n,k,j})$ such that $\widehat{a}_{n,k,i} \xrightarrow{w^*} G_{n,k}$ and $\widehat{b}_{n,k,j} \xrightarrow{w^*} H_{n,k}$. Thus $\widehat{a} =$ weak*-norm-weak*-$\lim(\widehat{a}_{n,k,i} \cdot \alpha_n)\Box\widehat{b}_{n,k,j}$. Hence $\widehat{a}$ belong to weak*-closure $(\mathcal{A}\mathfrak{A}\mathcal{A})$, and so a is in weak closure of $(\mathcal{A}\mathfrak{A}\mathcal{A})$. Therefore a is in the weak closure of span $(\mathcal{A}\mathfrak{A}\mathcal{A})$. Now the result follows from convexity of span $(\mathcal{A}\mathfrak{A}\mathcal{A})$. □

We say the Banach algebra $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from left (right) if for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$, $\alpha \cdot a = f(\alpha)a$ $(a \cdot \alpha = f(\alpha)a)$, where f is a continuous linear functional on $\mathfrak{A}$. Now if $\mathfrak{A}$ acts on $\mathcal{A}$ trivially from left and $\mathcal{A}/J$ has a bounded approximate identity, then for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$ we have $f(\alpha)a - \alpha \cdot a \in J$ [28, lemma 5.8].

Proposition 3.7. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with trivial left action and $\mathcal{A}/J$ be a commutative $\mathfrak{A}$-module (or $\mathcal{A}/J$ has a bounded approximate identity) and also $\mathcal{A}^{**}/N$ be commutative $\mathfrak{A}$-module (or N is w^*-closed). If $\mathcal{A}^{**}$ is weakly module amenable, then $\mathcal{A}^{**}/J^{\perp\perp}$ is weakly amenable.*

Proof. Assume that $D : \mathcal{A}^{**}/J^{\perp\perp} \longrightarrow (\mathcal{A}^{**}/J^{\perp\perp})^*$ is a derivation. The image of $G\Box F$ in $(\mathcal{A}^{**}/J^{\perp\perp}, \Box)$ is $G\Box F + J^{\perp\perp}$. Obviously, $\mathcal{A}^{**}/N$ is a $\mathcal{A}^{**}$-bimodule, with module actions given by

$$G \cdot (F + N) := G\Box F + J^{\perp\perp}, \quad (F + N) \cdot G := F\Box G + J^{\perp\perp}, \quad (F, G \in \mathcal{A}^{**}).$$

Let N be w*-closed. Since $\mathcal{A}/J$ is commutative $\mathfrak{A}$-module, then it follows from the proof of Theorem 2.2 that $\mathcal{A}^{**}/N$ is a commutative $\mathfrak{A}$-module. Consider $\tilde{D} : \mathcal{A}^{**} \longrightarrow (\mathcal{A}^{**}/N)^*$, defined by $\langle \tilde{D}(F), G+N\rangle := \langle D(F+J^{\perp\perp}), G+J^{\perp\perp}\rangle$ $(F, G \in \mathcal{A}^{**})$. It is easy to show that, for F and G in $\mathcal{A}^{**}$, we have

$$\tilde{D}(F \pm G) = \tilde{D}(F) \pm \tilde{D}(G), \quad \tilde{D}(F\Box G) = \tilde{D}(F) \cdot G + F \cdot \tilde{D}(G).$$

Also for $\alpha \in \mathfrak{A}$, we have

$$\begin{aligned}
\langle \tilde{D}(F \cdot \alpha), G + N\rangle &= \langle D(F \cdot \alpha + J^{\perp\perp}), G + J^{\perp\perp}\rangle \\
&= \langle D(f(\alpha)F + J^{\perp\perp}), G + J^{\perp\perp}\rangle \\
&= \langle \tilde{D}(F) \cdot \alpha, G + N\rangle.
\end{aligned}$$

Note that we have used Lemma 3.3 in the second equality. On the other hand, since the left $\mathfrak{A}$-module action on $\mathcal{A}$ is trivial, $\tilde{D}(\alpha \cdot F) = \alpha \cdot \tilde{D}(F)$. Hence $\tilde{D}$ is a module derivation. Therefore there exist $\Phi \in (\mathcal{A}^{**}/N)^*$ such that $\tilde{D}(F) = F \cdot \Phi - \Phi \cdot F$, for all $F \in \mathcal{A}^{**}$. Consider the canonical embedding $j : (\mathcal{A}/J)^* \longrightarrow (\mathcal{A}^{**}/J^{\perp\perp})^*$. Put $\Psi = j \circ \phi^*(\Phi)$, then for each F and G in $\mathcal{A}^{**}$, we have

$$\begin{aligned}
\langle D(F + J^{\perp\perp}), G + J^{\perp\perp}\rangle &= \langle \tilde{D}(F), G + N\rangle \\
&= \langle F \cdot \Phi - \Phi \cdot F, G + N\rangle \\
&= \langle \Phi, G\Box F + F\Box G + N\rangle \\
&= \langle \phi^*(\Phi), \lambda(G\Box F + F\Box G + N)\rangle \\
&= \langle \lambda(G\Box F + F\Box G + N), \Phi \circ \phi\rangle \\
&= \langle j(\Phi \circ \phi), G\Box F + F\Box G + J^{\perp\perp}\rangle \\
&= \langle \Psi, G\Box F + F\Box G + J^{\perp\perp}\rangle \\
&= \langle (F + J^{\perp\perp}) \cdot \Psi - \Psi \cdot (F + J^{\perp\perp}), G + J^{\perp\perp}\rangle.
\end{aligned}$$

□

Corollary 3.8. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with trivial left action and $\mathcal{A}/J$ be a commutative $\mathfrak{A}$-module. If one of the following conditions hold* *(i) $\mathcal{A}/J$ is a left ideal in $\mathcal{A}^{**}/J^{\perp\perp}$.*

(ii) $\mathcal{A}/J$ is a dual Banach algebra.

(iii) $\mathcal{A}/J$ is Arens regular and every derivation from $\mathcal{A}/J$ into $J^{\perp}$ is weakly compact.

Then weak module amenability $\mathcal{A}^{**}$ implies weak module amenability $\mathcal{A}$.

Proof. This is a consequence of Propositions 3.4 and 3.7, and [17, Theorem 2.3], [16, Theorem 2.2], and [13, Corollary 7.5] for the parts (i), (ii), and (iii), respectively. □

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Massoud Amini
Department of Mathematics
Tarbiat Modares University
Tehran 14115-175, Iran
Email: `mamini@modares.ac.ir`

Abasalt Bodaghi
Department of Mathematics
Islamic Azad University
Garmsar Branch, Garmsar, Iran
Email: `abasalt.bodaghi@gmail.com`