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Invertible Matrices over Idempotent Semirings

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Abstract: By an *idempotent semiring* we mean a commutative semiring $(S, +, \cdot)$ with zero 0 and identity 1 such that $x + x = x = x^2$ for all $x \in S$. In 1963, D.E. Rutherford showed that a square matrix A over an idempotent semiring S of 2 elements is invertible over S if and only if A is a permutation matrix. By making use of C. Reutenauer and H. Straubing's theorems, we extend this result to an idempotent semiring as follows: A square matrix A over an idempotent semiring S is invertible over S if and only if the product of any two elements in the same column [row] is 0 and the sum of all elements in each row [column] is 1.

Keywords: Idempotent semiring, invertible matrix

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1 Introduction

A *semiring* is a system $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are semigroups and $x\cdot(y+z) = x\cdot y+x\cdot z$ and $(y+z)\cdot x = y\cdot x+z\cdot x$ for all $x, y, z \in S$. A semiring $(S, +, \cdot)$ is called *additively* [*multiplicatively*] *commutative* if $x + y = y + x$ $[x \cdot y = y \cdot x]$ for all $x, y \in S$ and it is called *commutative* if it is both additively and multiplicatively commutative. An element $0 \in S$ is called a *zero* of $(S, +, \cdot)$ if $x + 0 = 0 + x = x$

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and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$. By an *identity* of a semiring $(S, +, \cdot)$ we mean an element $1 \in S$ such that $x \cdot 1 = 1 \cdot x = x$ for all $x \in S$. Notice that a zero and an identity of a semiring are unique.

By an idempotent semiring we mean a commutative semiring with zero 0 and identity 1 such that $x + x = x = x^2$ for all $x \in S$.

Example 1.1. Let $S \subseteq [0,1]$ be such that $0, 1 \in S$. Define the operations $\oplus$ and $\odot$ on S by

$$x \oplus y = \max\{x, y\} \quad \text{and} \quad x \odot y = \min\{x, y\} \ \text{ for all } x, y \in S.$$

Then $(S, \oplus, \odot)$ is an idempotent semiring having 0 and 1 as its zero and identity, respectively and it may be written as $(S, \max, \min)$.

Example 1.2. Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. Define

$$A \oplus B = A \cup B \quad \text{and} \quad A \odot B = A \cap B \ \text{ for all } A, B \in \mathcal{P}(X).$$

Then $(\mathcal{P}(X), \oplus, \odot)$ is an idempotent semiring having $\emptyset$ and X as its zero and identity, respectively and it may be written as $(\mathcal{P}(X), \cup, \cap)$.

Let S be a commutative semiring with zero 0 and identity 1, n a positive integer and $M_n(S)$ the set of all $n \times n$ matrices over S. Then under usual addition and multiplication, $M_n(S)$ is an additively commutative semiring. The $n \times n$ zero matrix 0_n and the $n \times n$ identity matrix I_n over S are the zero and the identity of the semiring $M_n(S)$, respectively. If S contains more than one element and $n > 1$, then $M_n(S)$ is not multiplicatively commutative. For $A \in M_n(S)$ and $i, j \in \{1, \ldots, n\}$, let A_{ij} be the element (entry) of A in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column. The transpose of $A \in M_n(S)$ will be denoted by A^t, that is, $A^t_{ij} = A_{ji}$ for all $i, j \in \{1, \ldots, n\}$. Then for $A, B \in M_n(S)$, $(A^t)^t = A, (A + B)^t = A^t + B^t$ and $(AB)^t = B^t A^t$. A matrix $A \in M_n(S)$ is said to be *invertible* over S if $AB = BA = I_n$ for some $B \in M_n(S)$. Notice that such B is unique and B is called the *inverse* of A. Also, A is invertible over S if and only if A^t is invertible over S. A matrix $A \in M_n(S)$ is called a *permutation matrix* if every element (entry) of A is either 0 or 1 and each row and each column contains exactly one 1. A permutation matrix A over S is clearly invertible over S and A^t is the inverse of A. In 1963, D.E. Rutherford characterized the invertible matrices in $M_n(S)$ where S is an idempotent semiring of 2 elements as follows:

Theorem 1.3. [2] *Let S be an idempotent semiring of 2 elements. Then a square matrix A over S is invertible over S if and only if A is a permutation matrix.*

Let $\mathcal{S}_n$ be the symmetric group of degree $n \geq 2$, $\mathcal{A}_n$ the alternating group of degree n and $\mathcal{B}_n = \mathcal{S}_n \smallsetminus \mathcal{A}_n$, that is,

$$\mathcal{A}_n = \{\, \sigma \in \mathcal{S}_n \mid \sigma \text{ is an even permutation} \,\},$$
$$\mathcal{B}_n = \{\, \sigma \in \mathcal{S}_n \mid \sigma \text{ is an odd permutation} \,\}.$$

If S is a commutative semiring with zero and identity and n is a positive integer greater than 1, then the *positive determinant* and the *negative determinant* of $A \in M_n(S)$ are defined respectively by

$$\det{}^+ A = \sum_{\sigma \in \mathcal{A}_n} \left(\prod_{i=1}^{n} A_{i\sigma(i)} \right),$$
$$\det{}^- A = \sum_{\sigma \in \mathcal{B}_n} \left(\prod_{i=1}^{n} A_{i\sigma(i)} \right).$$

Notice that $\det^+ I_n = 1$ and $\det^- I_n = 0$. In 1984, C. Reutenauer and H. Straubing [1] gave the following significant results.

Theorem 1.4. ([1]) *Let S be a commutative semiring with zero and identity and n a positive integer ≥ 2. If $A, B \in M_n(S)$, then there is an element $r \in S$ such that*

$$\det{}^+(AB) = (\det{}^+ A)(\det{}^+ B) + (\det{}^- A)(\det{}^- B) + r,$$
$$\det{}^-(AB) = (\det{}^+ A)(\det{}^- B) + (\det{}^- A)(\det{}^+ B) + r.$$

Theorem 1.5. ([1]) *Let S be a commutative semiring with zero and identity and n a positive integer. For $A, B \in M_n(S)$, if $AB = I_n$, then $BA = I_n$.*

The purpose of this paper is to extend Theorem 1.3 to an idempotent semiring by making use of Theorem 1.4 and Theorem 1.5. We show that an $n \times n$ matrix over an idempotent semiring S with zero 0 and identity 1 is invertible over S if and only if

(i) the product of any two elements in the same column [row] is 0 and

(ii) the sum of all elements in each row [column] is 1.

2 Invertible Matrices over Idempotent Semirings

The following series of lemmas is needed. The first one is evident.

Lemma 2.1. *Let S be an idempotent semiring with zero 0 and identity 1. Then the following statements hold.*

(i) *For $x, y \in S$, $x + y = 0 \Rightarrow x = 0 = y$.*

(ii) *For $x, y \in S$, $xy = 1 \Rightarrow x = 1 = y$.*

Lemma 2.2. *Let S be an idempotent semiring and n a positive integer ≥ 2. If $A \in M_n(S)$ is invertible over S, then $\det^+ A + \det^- A = 1$.*

Proof. Let $B \in M_n(S)$ be such that $AB = BA = I_n$. By Theorem 1.4, there exists an element $r \in S$ such that

$$\det{}^+(AB) = (\det{}^+A)(\det{}^+B) + (\det{}^-A)(\det{}^-B) + r,$$
$$\det{}^-(AB) = (\det{}^+A)(\det{}^-B) + (\det{}^-A)(\det{}^+B) + r.$$

But $\det^+(AB) = \det^+ I_n = 1$ and $\det^-(AB) = \det^- I_n = 0$, so we have that

$$1 = (\det{}^+A)(\det{}^+B) + (\det{}^-A)(\det{}^-B) + r,$$
$$0 = (\det{}^+A)(\det{}^-B) + (\det{}^-A)(\det{}^+B) + r.$$

The last equality and Lemma 2.1(i) yield the result that

$$(\det{}^+A)(\det{}^-B) = (\det{}^-A)(\det{}^+B) = r = 0.$$

Then

$$\begin{aligned} 1 &= (\det{}^+A)(\det{}^+B) + (\det{}^-A)(\det{}^-B) \\ &= (\det{}^+A)(\det{}^+B) + (\det{}^-A)(\det{}^-B) + (\det{}^+A)(\det{}^-B) + (\det{}^-A)(\det{}^+B) \\ &= (\det{}^+A + \det{}^-A)(\det{}^+B + \det{}^-B). \end{aligned}$$

By Lemma 2.1(ii), we have that $\det^+ A + \det^- A = 1$, as desired. □

Theorem 2.3. *Let S be an idempotent semiring with zero 0 and identity 1, n a positive integer and $A \in M_n(S)$. Then A is invertible over S if and only if*

(i) *the product of any two elements in the same column is 0 and*

(ii) *the sum of all elements in each row is 1.*

Proof. By Lemma 2.1(ii), the theorem is obviously true for $n = 1$.

Let $n > 1$ and assume that A is invertible over S. Let $B \in M_n(S)$ be such that $AB = BA = I_n$. Let $i, j \in \{1, \ldots, n\}$ be distinct. Then

$$0 = (I_n)_{ij} = (AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

It follows from Lemma 2.1(i) that $A_{ik}B_{kj} = 0$ for all $k \in \{1, \ldots, n\}$. This proves that

$$A_{lk}B_{kt} = 0 \text{ for all } l, t, k \in \{1, \ldots, n\} \text{ such that } l \neq t. \tag{1}$$

Then for $k \in \{1, \ldots, n\}$,

$$\begin{aligned} A_{ik}A_{jk} &= (A_{ik}A_{jk})1 \\ &= (A_{ik}A_{jk})(BA)_{kk} \\ &= A_{ik}A_{jk}\big(\sum_{t=1}^{n} B_{kt}A_{tk}\big) \\ &= \sum_{t=1}^{n} A_{ik}A_{jk}B_{kt}A_{tk} \\ &= A_{ik}A_{jk}B_{kj}A_{jk} + \sum_{\substack{t=1 \\ t \neq j}}^{n} A_{ik}A_{jk}B_{kt}A_{tk} \\ &= (A_{ik}B_{kj})A_{jk} + \sum_{\substack{t=1 \\ t \neq j}}^{n} A_{ik}(A_{jk}B_{kt})A_{tk} \\ &= 0 + 0 = 0 \qquad \text{from (1).} \end{aligned}$$

Hence (i) is proved.

From Lemma 2.2, we have that $\det^+ A + \det^- A = 1$. By (i), we have that

$$A_{1k_1}A_{2k_2} \ldots A_{nk_n} = 0 \text{ if } k_1, \ldots, k_n \in \{1, \ldots, n\} \text{ are not all distinct.} \tag{2}$$

Then

$$\begin{aligned} \big(\sum_{k=1}^{n} A_{1k}\big)\big(\sum_{k=1}^{n} A_{2k}\big) \ldots \big(\sum_{k=1}^{n} A_{nk}\big) &= \sum_{k_1, \ldots, k_n \in \{1, \ldots, n\}} A_{1k_1}A_{2k_2} \ldots A_{nk_n} \\ &= \sum_{\sigma \in \mathcal{S}_n} A_{1\sigma(1)}A_{2\sigma(2)} \ldots A_{n\sigma(n)} \quad \text{from (2)} \\ &= \det^+ A + \det^- A = 1. \end{aligned}$$

Hence by Lemma 2.1(ii), $\sum_{k=1}^{n} A_{1k} = \sum_{k=1}^{n} A_{2k} = \cdots = \sum_{k=1}^{n} A_{nk} = 1$. Therefore (ii) is proved.

Conversely, assume that (i) and (ii) hold. Claim that $AA^t = I_n$. If $i \in \{1, \ldots, n\}$, then from (ii),

$$(AA^t)_{ii} = \sum_{k=1}^{n} A_{ik}A^t_{ki} = \sum_{k=1}^{n} A_{ik}A_{ik} = \sum_{k=1}^{n} A_{ik} = 1.$$

Also, for distinct $i, j \in \{1, \ldots, n\}$, from (i)

$$(AA^t)_{ij} = \sum_{k=1}^{n} A_{ik}A^t_{kj} = \sum_{k=1}^{n} A_{ik}A_{jk} = 0.$$

This shows that $AA^t = I_n$. Therefore by Theorem 1.5, $A^tA = I_n$. Hence A is invertible over S. □

Since A is invertible over S if and only if A^t is invertible over S, the following result is obtained directly from Theorem 2.3.

Corollary 2.4. *Let S be an idempotent semiring with zero 0 and identity 1, n a positive integer and $A \in M_n(S)$. Then A is invertible over S if and only if*

(i) *the product of any two elements in the same row is 0 and*

(ii) *the sum of all elements in each column is 1.*

Corollary 2.5. *Let $(S, \max, \min)$ be the idempotent semiring defined as in Example 1.1, n a positive integer and $A \in M_n(S)$. Then A is invertible over S if and only if A is a permutation matrix.*

Proof. Assume that A is invertible over S. Let $i \in \{1, \ldots, n\}$. If there are distinct $l, t \in \{1, \ldots, n\}$ such that $A_{il} \neq 0$ and $A_{it} \neq 0$, then $A_{il} \odot A_{it} = \min\{A_{il}, A_{it}\} \neq 0$ which is contrary to Corollary 2.4. By Theorem 2.3, $A_{i1} \oplus \cdots \oplus A_{in} = 1$. Then $1 = \max\{A_{i1}, \ldots, A_{in}\}$, so $A_{ik} = 1$ for some $k \in \{1, \ldots, n\}$. This shows that every row of A contains only one nonzero element which is 1. We can show similarly by Theorem 2.3 and Corollary 2.4 that every column of A contains only one nonzero element which is 1. Hence A is a permutation matrix.

As mentioned previously, a permutation matrix is invertible. □

Remark 2.6. From the proof of Theorem 2.3, we have that if $A \in M_n(S)$ is invertible over S, then the inverse of A is A^t. It can be seen that Theorem 1.3 is a special case of Corollary 2.5.

Example 2.7. Let $(\mathcal{P}(X), \cup, \cap)$ be the idempotent semiring defined in Example 1.2. Let $X_1, X_2, \ldots, X_n$ be subsets of X such that $X = X_1 \cup X_2 \cup \ldots \cup X_n$ and $X_1, \ldots, X_n$ are pairwise disjoint. By Theorem 2.3,

$$A = \begin{bmatrix} X_1 & X_2 & \cdots & X_{n-1} & X_n \\ X_n & X_1 & \cdots & X_{n-2} & X_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ X_3 & X_4 & \cdots & X_1 & X_2 \\ X_2 & X_3 & \cdots & X_n & X_1 \end{bmatrix} \in M_n((\mathcal{P}(X), \cup, \cap))$$

is invertible over $(\mathcal{P}(X), \cup, \cap)$. By Remark 2.6, the inverse of A is

$$\begin{bmatrix} X_1 & X_n & \cdots & X_3 & X_2 \\ X_2 & X_1 & \cdots & X_4 & X_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{n-1} & X_{n-2} & \cdots & X_1 & X_n \\ X_n & X_{n-1} & \cdots & X_2 & X_1 \end{bmatrix}.$$

References

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