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On the McShane Integral in Topological Vector Spaces

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Abstract: In this paper, the McShane integral of a function with values in a topological vector space (TVS) is defined. Results such as uniqueness of the integral, the Cauchy criterion for integrability, integrability of a continuous function, linearity of the integral, and additive property on subintervals are presented. Moreover, this study shows that the TVS-version and the Banach-version of the McShane integral are equivalent whenever the space under consideration is Banach.

Keywords: McShane integral, topological vector space, Banach space

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1 Preliminary Concepts

The Lebesgue integral has been considered by a lot of mathematicians as the official or standard integral in mathematical research. This integral overcomes some of the defects of the Riemann integral. However, the problem with Lebesgue's integral is that it is not easy to understand. One has to master a considerable amount of measure theory before he can fully understand it. Also, such an integral does not inherit the naturalness of the Riemann integral.

In the late 1960's, E.J. McShane defined a Riemann-type integral and proved that this integral is equivalent to the Lebesgue integral. As a Riemann-type integral, it is simpler to handle than the Lebesgue integral. Further, his integral

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does not involve concepts such as σ-algebras and measures. Several extensions of the McShane integral had been done. Gordon extended the definition to Banach-valued functions while Canoy extended the definition to functions with values in a ranked countably normed spaces.

In this study, we will extend the definition of the McShane integral to functions with values in a topological vector space.

Let X be a Hausdorff topological space. We say that X is a **topological vector space** if X is a real vector space and the operations, vector addition and scalar multiplication, are continuous.

The next lemma (see [6, pp.94-95]) is a property of a topological vector space that we will use in this paper.

Lemma 1.1. *Let X be a topological vector space. Then there is a local base $\mathcal{B}$ of θ (the zero vector), satisfying the following:*

1. *If $U, V \in \mathcal{B}$, then there is a $W \in \mathcal{B}$ with $W \subseteq U \cap V$.*
2. *If $U \in \mathcal{B}$ and $x \in U$, there is a $V \in \mathcal{B}$ such that $x + V \subseteq U$.*
3. *If $U \in \mathcal{B}$, there is a $V \in \mathcal{B}$ such that $V + V \subseteq U$.*
4. *If $U \in \mathcal{B}$ and $x \in X$, then there is $k \in \mathbf{R}$ such that $x \in kU$.*
5. *If $U \in \mathcal{B}$ and $0 < |k| \leq 1$, then $kU \subseteq U$ and $kU \in \mathcal{B}$.*
6. $\bigcap\{U : U \in \mathcal{B}\} = \{\theta\}$.

Conversely, given a collection $\mathcal{B}$ of subsets containing θ and satisfying the above conditions, there is a topology for X making X a topological vector space and having $\mathcal{B}$ as a local base at θ.

By a θ-nbd, we mean an open set containing θ.

Let X be a nonempty set. A family $\mathcal{F} = \{A_\delta : \delta \in \Delta\}$ of subsets of X is a **filter** in X if the following are satisfied:

i. For every $\delta \in \Delta$, $A_\delta \neq \emptyset$.

ii. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

iii. If $A \in \mathcal{F}$, $B \subseteq X$, and $A \subseteq B$, then $B \in \mathcal{F}$.

Let X be a topological vector space and $\mathcal{F} = \{A_\delta : \delta \in \Delta\}$ be a filter in X and $x \in X$. We say that $\mathcal{F}$ **converges** to x, written $\mathcal{F} \to x$, if for every θ-nbd U there exists $A \in \mathcal{F}$ such that $A - x \subseteq U$. In this case we say that $\mathcal{F}$ is a **convergent filter**. We say that $\mathcal{F}$ is **Cauchy** if for every θ-nbd U there exists $A \in \mathcal{F}$ such that $A - A \subseteq U$.

Let X be a topological vector space. We say that X is **complete** if every Cauchy filter in X converges. We say that X is **locally convex** if there is a local base at θ whose members are convex.

Let δ be a positive function on the closed interval $[a,b]$. We say that a collection $P = \{([x_{i-1}, x_i], t_i) : 1 \le i \le n\}$ is a **δ-fine free tagged partition** of $[a,b]$ if $\{[x_{i-1}, x_i] : 1 \le i \le n\}$ is a partition of $[a,b]$ and $t_i \in [a,b]$ and $[x_{i-1}, x_i] \subseteq (t_i - \delta(t_i), t_i + \delta(t_i))$, for every i, where $1 \le i \le n$.

2 Results

Definition 2.1. A function $f : [a,b] \to X$ is **McShane integrable** on $[a,b]$, if there is an $\alpha \in X$ such that for any θ-nbd U there is a positive function δ on $[a,b]$ such that whenever a δ-fine free tagged partition $P = \{([x_{i-1}, x_i], t_i) : 1 \le i \le n\}$ of $[a,b]$ is given, we have $S(f;P) - \alpha \in U$, where $S(f;P) = \sum_{i=1}^{n} (x_i - x_{i-1}) f(t_i)$.

The following theorem guarantees the uniqueness of α, if it exists. We call α the **McShane integral** of f in $[a,b]$ and write $(M)\int_a^b f = \alpha$.

Theorem 2.2. *A given function $f : [a,b] \to X$ has at most one McShane integral on $[a,b]$.*

Proof. Suppose that f is McShane integrable on $[a,b]$. Suppose further that α_1, α_2 are the McShane integrals of f with $\alpha_1 \ne \alpha_2$. Let U_1 and U_2 be disjoint neighborhoods of α_1 and α_2, respectively. Then, $U_1 - \alpha_1$ and $U_2 - \alpha_2$ are θ-nbds. Let $V_1 = U_1 - \alpha_1$ and $V_2 = U_2 - \alpha_2$. Then, there exist positive functions δ_1 and δ_2 such that for every δ_1-fine free tagged partition P_1 and every δ_2-fine free tagged partition P_2 of $[a,b]$, we have $S(f;P_1) - \alpha_1 \in V_1$ and $S(f;P_2) - \alpha_2 \in V_2$. Let $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$ for all $x \in [a,b]$, and P be a δ-fine free tagged partition of $[a,b]$. Clearly, P is δ_1-fine and δ_2-fine. Thus, $S(f;P) - \alpha_1 \in V_1$ and $S(f;P) - \alpha_2 \in V_2$. It follows that $S(f;P) \in V_1 + \alpha_1 = U_1$ and $S(f;P) \in V_2 + \alpha_2 = U_2$. This is a contradiction. Therefore, $\alpha_1 = \alpha_2$. $\square$

In what follows, $M([a,b],X)$ denotes the set of all McShane-integrable X-valued functions on $[a,b]$.

We now present the some properties of the McShane integral. First, we show the linearity of the integral.

Theorem 2.3. *Let X be a topological vector space. If k is a real number and $f,g \in M([a,b],X)$, then $kf, f+g \in M([a,b],X)$ with*

$$(M)\int_a^b kf \;=\; k(M)\int_a^b f$$

and

$$(M)\int_a^b (f+g) \;=\; (M)\int_a^b f + (M)\int_a^b g.$$

Proof. Let $(M)\int_a^b f = \alpha$. The case $k=0$ is obvious. Suppose $k \neq 0$. Let U be a θ-nbd. Then, there exists a positive function δ such that for any δ-fine free tagged partition P, we have, $S(f;P) - \alpha \in \frac{1}{k}U$. Thus,

$$S(kf;P) - k\alpha = kS(f;P) - k\alpha = k\left(S(f;P) - \alpha\right) \in U.$$

Hence, $kf \in M([a,b],X)$ and $(M)\int_a^b kf = k(M)\int_a^b f$.

Let $(M)\int_a^b f = \alpha_1$ and $(M)\int_a^b g = \alpha_2$. Let U be a θ-nbd. Then, there exists a θ-nbd V such that $V+V \subseteq U$. Consequently, there exists a positive function δ_1 on $[a,b]$ such that for every δ_1-fine free tagged partition P_1, we have, $S(f;P_1) - \alpha_1 \in V$. Also, there exists a positive function δ_2 on $[a,b]$ such that for every δ_2-fine free tagged partition P_2, we have, $S(g;P_2) - \alpha_2 \in V$. Let $\delta(t) = \min\{\delta_1(t), \delta_2(t)\}$ for all $t \in [a,b]$ and P be a δ-fine free tagged partition on $[a,b]$. Then, P is also δ_1-fine and δ_2-fine. Note that $S(f+g;P) = S(f;P) + S(g;P)$. Thus,

$$S(f+g;P) - (\alpha_1 + \alpha_2) = (S(f;P) - \alpha_1) + (S(g;P) - \alpha_2) \in V + V \subseteq U.$$

Hence, $f+g \in M([a,b],X)$ and

$$(M)\int_a^b (f+g) \;=\; (M)\int_a^b f + (M)\int_a^b g.$$

□

The next theorem shows that the integral satisfies the additive property on subintervals.

Theorem 2.4. *Let X be a topological vector space. If $f \in M([a,b],X)$ and $f \in M([b,c],X)$, then $f \in M([a,c],X)$ and*

$$(M)\int_a^c f \;=\; (M)\int_a^b f + (M)\int_b^c f.$$

Proof. Let $(M)\int_a^b f = \alpha_1$ and $(M)\int_b^c f = \alpha_2$. Let U be a θ-nbd. Then, there exists a θ-nbd V such that $V+V \subseteq U$. Thus, there exists a positive function δ_1 such that for every δ_1-fine free tagged partition P_1 of $[a,b]$, we have, $S(f;P_1) - \alpha_1 \in V$. Also, there exists a positive function δ_2 such that for every δ_2-fine free tagged partition P_2 of $[b,c]$, we have, $S(f;P_2) - \alpha_2 \in V$. Let

$$\delta(t) = \begin{cases} \min\{\delta_1(t), b-t\}, & \text{if } a \le t < b \\ \min\{\delta_1(t), \delta_2(t)\}, & \text{if } t = b \\ \min\{\delta_2(t), t-b\}, & \text{if } b < t \le c. \end{cases}$$

Let P be a δ-fine free tagged partition of $[a,c]$. It is easy to show that $S(f;P) = S(f;P_1) + S(f;P_2)$ for some δ_1-fine free tagged partition P_1 of $[a,b]$ and δ_2-fine free tagged partition P_2 of $[b,c]$. Hence,

$$S(f;P) - (\alpha_1+\alpha_2) = (S(f;P_1) - \alpha_1) + (S(f;P_2) - \alpha_2) \in V + V \subseteq U.$$

This proves the theorem. □

In the next theorem, we give the Cauchy criterion for integrability.

Theorem 2.5 (Cauchy criterion). *Let X be a complete topological vector space. Then, $f \in M([a,b],X)$ if and only if for every θ-nbd U there exists a positive function δ on $[a,b]$ such that for any δ-fine free tagged partitions P_1 and P_2 of $[a,b]$, we have $S(f;P_1) - S(f;P_2) \in U$.*

Proof. Suppose $(M)\int_a^b f = \alpha$. Let U be a θ-nbd. Then, there exists a θ-nbd V such that $V - V \subseteq U$. Thus, there exists a positive function δ on $[a,b]$ such that if P is a δ-fine free tagged partition of $[a,b]$, we have $S(f;P) - \alpha \in V$. Hence, if P_1 and P_2 are δ-fine free tagged partitions of $[a,b]$,

$$S(f;P_1) - S(f;P_2) = (S(f;P_1) - \alpha) - (S(f;P_2) - \alpha) \in V - V \subseteq U.$$

For each positive function δ on $[a,b]$, let

$$A_\delta = \{S(f;P):\ P \text{ is a } \delta\text{-fine free tagged partition of } [a,b]\}.$$

Let $\mathcal{D} = \{A_\delta : \delta$ is a positive function on $[a,b]\}$. Clearly, $\mathcal{D}$ is a filterbase in X. Since X is complete, $\mathcal{D} \to \alpha$ for some $\alpha \in X$. We now show that α is the McShane integral of f.

Let U be a θ-nbd. Since $\mathcal{D} \to \alpha$, there exists a positive function δ on $[a,b]$ such that $A_\delta - \alpha \subseteq U$. Thus, if P is a δ-fine free tagged partition on $[a,b]$, we have, $S(f;P) - \alpha \in U$. Hence, $f \in M([a,b],X)$. □

We now apply the Cauchy criterion in showing the integrability on a subinterval.

Theorem 2.6. *Let X be a complete topological vector space. If $f \in M([a,b],X)$, then $f \in M([c,d],X)$ for every $[c,d] \subseteq [a,b]$.*

Proof. Let U be a θ-nbd. By Cauchy criterion, there exists a positive function δ on $[a,b]$ such that for every δ-fine free tagged partitions P and Q of $[a,b]$, we have, $S(f;P) - S(f;Q) \in U$. Let P_1 and Q_1 be δ-fine free tagged partitions of $[c,d]$.

Suppose $a = c$ and $d < b$. Let P_2 be a δ-fine free tagged partition of $[d,b]$. Clearly, $P = P_1 \cup P_2$ and $Q = Q_1 \cup P_2$ are δ-fine free tagged partitions of $[a,b]$. Hence,

$$\begin{aligned} S(f;P_1) - S(f;Q_1) &= [S(f;P_1) + S(f;P_2)] - [S(f;Q_1) + S(f;P_2)] \\ &= S(f;P) - S(f;Q) \in U. \end{aligned}$$

Suppose $a < c$ and $b = d$. Let P_2 be a δ-fine free tagged partition of $[a,c]$. Clearly, $P = P_1 \cup P_2$ and $Q = Q_1 \cup P_2$ are δ-fine free tagged partitions of $[a,b]$. Hence,

$$\begin{aligned} S(f;P_1) - S(f;Q_1) &= [S(f;P_1) + S(f;P_2)] - [S(f;Q_1) + S(f;P_2)] \\ &= S(f;P) - S(f;Q) \in U. \end{aligned}$$

Suppose $a < c$ and $d < b$. Let P_2 be a δ-fine free tagged partition of $[a,c]$ and P_3 be a δ-fine free tagged partition of $[d,b]$. Clearly, $P = P_1 \cup P_2 \cup P_3$ and $Q = Q_1 \cup P_2 \cup P_3$ are δ-fine free tagged partitions of $[a,b]$. Hence,

$$\begin{aligned} S(f;P_1) - S(f;Q_1) &= [S(f;P_1) + S(f;P_2) + S(f;P_3)] \\ &\quad - [S(f;Q_1) + S(f;P_2) + S(f;P_3)] \\ &= S(f;P) - S(f;Q) \in U. \end{aligned}$$

Hence, in any case, $S(f;P_1) - S(f;Q_1) \in U$.

Therefore, by Cauchy criterion, $f \in M([c,d],X)$. □

Let $A \subseteq \mathbf{R}$ and $a \in A$. A function $f : A \to X$ is **continuous at** a if for every θ-nbd U there exists $\delta > 0$ such that for any $x \in A$ with $|x - a| < \delta$, we have $f(x) - f(a) \in U$. We say that f is **continuous on** A if it is continuous at each point of A. We say that f is **uniformly continuous** if for every θ-nbd U there exists a $\delta > 0$ such that for any $x, y \in A$ with $|x - y| < \delta$, we have $f(x) - f(y) \in U$.

Apparently, every uniformly continuous function is continuous. The converse is not true. However, these two concepts are equivalent when the domain is $[a,b]$. The next lemma gives us something more.

Lemma 2.7. *Let X be a topological vector space, $f : [a,b] \to X$, and $A \subseteq [a,b]$. If A is compact and f is continuous on A, then for every θ-nbd U there exists a $\delta > 0$ such that for any $x \in A$ and $y \in [a,b]$ with $|x - y| < \delta$, we have $f(x) - f(y) \in U$.*

Proof. Let U be a θ-nbd and V be a θ-nbd such that $V - V \subseteq U$. Since f is continuous on A, for every $t \in A$ there exists a $\delta(t) > 0$ such that $f(x) - f(t) \in V$ whenever $x \in [a,b]$ and $|x - t| < \delta(t)$. Note that,

$$\mathcal{C} = \left\{ \left(t - \frac{\delta(t)}{2}, t + \frac{\delta(t)}{2} \right) : t \in A \right\}$$

covers A. Since A is compact, there exists a finite cover, say

$$\mathcal{C}' = \left\{ \left(t_k - \frac{\delta(t_k)}{2}, t_k + \frac{\delta(t_k)}{2} \right) : t_k \in A, 1 \le k \le n \right\}.$$

Let

$$\delta' = \min \left\{ \frac{\delta(t_k)}{2} : 1 \le k \le n \right\}.$$

Let $x \in A$ and $y \in [a,b]$ such that $|x - y| < \delta'$. Since $\mathcal{C}'$ covers A, there exists k such that

$$x \in \left(t_k - \frac{\delta(t_k)}{2}, t_k + \frac{\delta(t_k)}{2} \right).$$

This implies that

$$|x - t_k| < \frac{\delta(t_k)}{2} < \delta(t_k).$$

Accordingly, $f(x) - f(t_k) \in V$. Moreover,

$$|y - t_k| \leq |x - y| + |x - t_k| < \delta' + \frac{\delta(t_k)}{2} \leq \delta(t_k),$$

so it follows that $f(y) - f(t_k) \in V$. Hence,

$$f(x) - f(y) = (f(x) - f(t_k)) - (f(y) - f(t_k)) \in V - V \subset U.$$

This proves the lemma. □

We now show that every continuous function is McShane integrable when the range is complete and locally convex.

Theorem 2.8. *Let X be a complete and locally convex topological vector space and $f : [a,b] \to X$. If f is continuous on $[a,b]$, then $f \in M([a,b], X)$.*

Proof. Let U be a θ-nbd. Since X is locally convex, we can assume that U is convex. Since f is continuous, then by Lemma 2.7, f is uniformly continuous. Thus, there exists $\lambda > 0$ such that if $f(x) - f(y) \in \frac{1}{b-a}U$ whenever $x, y \in [a,b]$ and $|x - y| < \lambda$. Let $\delta(x) = \frac{\lambda}{2}$ for every $x \in [a,b]$. Let $P = \{(I_i, t_i) : 1 \leq i \leq n\}$ and $Q = \{(J_j, s_j) : 1 \leq j \leq m\}$ be δ-fine free tagged partitions on $[a,b]$. Then $S(f;P) = \sum_{i=1}^{n}\sum_{j=1}^{m}\ell(I_i \cap J_j)f(t_i)$, and $S(f;Q) = \sum_{j=1}^{m}\sum_{i=1}^{n}\ell(I_i \cap J_j)f(s_j)$. In effect,

$$S(f;P) - S(f;Q) = \sum_{i=1}^{n}\sum_{j=1}^{m}\ell(I_i \cap J_j)(f(t_i) - f(s_j)).$$

For each $i = 1, 2, \ldots, n$, let

$$Q_i = \{J \cap I_i : J \text{ is an interval in } Q \text{ and } \ell(I_i \cap J) \neq 0\}.$$

Clearly, Q_i is a partition of I_i. Let $T = \bigcup_{i=1}^{n} Q_i = \{H_k : 1 \leq k \leq p\}$. Clearly, T is a partition of $[a,b]$. Thus,

$$S(f;P) - S(f;Q) = \sum_{k=1}^{p}\ell(H_k)(f(u_k) - f(v_k)),$$

where $u_k = t_i$ and $v_k = s_j$ whenever $H_k = I_i \cap J_j$. Note that for each k, $|u_k - v_k| < \delta$. Thus, $f(u_k) - f(v_k) \in \frac{1}{b-a}U$, for each k. Hence, by convexity of U,

$$\begin{aligned} S(f;P) - S(f;Q) &\in \sum_{k=1}^{p}\ell(H_k)\left(\frac{1}{b-a}U\right) \\ &= \sum_{k=1}^{p}\left(\frac{\ell(H_k)}{b-a}U\right) \subseteq U. \end{aligned}$$

Therefore, by Cauchy criterion, $f \in M([a,b],X)$. □

The next result gives us a necessary and sufficient condition for integrability.

Theorem 2.9. *Let X be a complete topological vector space. Then, $f \in M([a,b], X)$ if and only if there is a function $F : [a,b] \to X$ satisfying the following condition: for every θ-nbd U there exists a positive function δ on $[a,b]$ such that if $P = \{([x_{i-1}, x_i], t_i) : 1 \le i \le n\}$ is a δ-fine free tagged partition of $[a,b]$ there exist open sets $U_1, U_2, \ldots, U_n$ with $\sum_{i=1}^n U_i \subseteq U$ such that $F(x_i) - F(x_{i-1}) - (x_i - x_{i-1})f(t_i) \in U_i$ for all i.*

Proof. Suppose $f \in M([a,b],X)$. Let $F(x) = (M)\int_a^x f$ where $(M)\int_a^a f = \theta$. Let U be a θ-nbd. Then, there exists a positive function δ on $[a,b]$ such that if $P = \{([x_{i-1}, x_i], t_i) : 1 \le i \le n\}$ is a δ-fine free tagged partition of $[a,b]$, $F(b) - S(f;P) \in U$. Note that

$$F(b) - S(f;P) = \sum_{i=1}^{n} \left(F(x_i) - F(x_{i-1}) - (x_i - x_{i-1})f(t_i)\right).$$

We now show that there exist open sets $U_1, U_2, \ldots, U_n$ with $\sum_{i=1}^n U_i \subseteq U$ such that $F(x_i) - F(x_{i-1}) - (x_i - x_{i-1})f(t_i) \in U_i$ for all i. The claim is obvious if $n = 1$. Suppose $n = 2$. Let

$$F(x_1) - F(x_0) - (x_1 - x_0)f(t_1) + F(x_2) - F(x_1) - (x_2 - x_1)f(t_2) \in U.$$

Since $+ : X \times X \to X$ is continuous, there exists an open set V in $X \times X$ containing $(F(x_1) - F(x_0) - (x_1 - x_0)f(t_1), F(x_2) - F(x_1) - (x_2 - x_1)f(t_2))$ such that $+(V) \subseteq U$. Since V is an open set in $X \times X$ containing

$$(F(x_1) - F(x_0) - (x_1 - x_0)f(t_1), F(x_2) - F(x_1) - (x_2 - x_1)f(t_2)),$$

there exist open sets U_1 and U_2 in X such that $F(x_1) - F(x_0) - (x_1 - x_0)f(t_1) \in U_1$, $F(x_2) - F(x_1) - (x_2 - x_1)f(t_2) \in U_2$, and $U_1 \times U_2 \subseteq V$. Note that

$$U_1 + U_2 = +(U_1 \times U_2) \subseteq +(V) \subseteq U.$$

This proves the claim for $n = 2$. Proceeding inductively, we obtain that for any n, there exist open sets $U_1, U_2, \ldots, U_n$ with $\sum_{i=1}^n U_i \subseteq U$ such that $F(x_i) - F(x_{i-1}) - (x_i - x_{i-1})f(t_i) \in U_i$ for all i.

Suppose there is a function $F : [a,b] \to X$ satisfying the stated condition. Let U be a θ-nbd. Then, there exists a positive function δ on $[a,b]$ such that if $P = \{([x_{i-1}, x_i], t_i) : 1 \leq i \leq n\}$ is a δ-fine free tagged partition of $[a, b]$ there exist neighborhoods $U_1, U_2, \ldots, U_n$ with $\sum_{i=1}^{n} U_i \subseteq U$ such that $F(x_i) - F(x_{i-1}) - (x_i - x_{i-1})f(t_i) \in U_i$ for all i. Thus,

$$\begin{aligned} F(b) - F(a) - S(f; P) &= \sum_{i=1}^{n} (F(x_i) - F(x_{i-1}) - (x_i - x_{i-1})f(t_i)) \\ &\in \sum_{i=1}^{n} U_i \subseteq U. \end{aligned}$$

Hence, $f \in M([a, b], X)$. □

The function F given in Theorem 2.9 is called an **M-primitive** of f in $[a, b]$.

The next example illustrates how we can use Theorem 2.9 in showing the McShane integrability of a function.

Example 2.10. Consider the Dirichlet function $f : [0, 1] \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1] \cap \mathbf{Q} \\ 0, & \text{if } x \in [0,1] \cap \mathbf{Q}^c. \end{cases}$$

Using Theorem 2.9, let us show that f is McShane integrable with M-primitive $F(x) = 0$, for all $x \in [0, 1]$. Let U be a 0-nbd. Then there exists $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subseteq U$. Since $[0, 1] \cap \mathbf{Q}$ is countable, there exists a bijective function $\alpha : [0, 1] \cap \mathbf{Q} \to \mathbf{N}$. Define δ as follows:

$$\delta(t) = \begin{cases} 1, & \text{if } t \in [0,1] \cap \mathbf{Q}^c \\ \frac{\epsilon}{2^{\alpha(t)+1}}, & \text{if } t \in [0,1] \cap \mathbf{Q}. \end{cases}$$

Let $P = \{([x_{i-1}, x_i], t_i) : 1 \leq i \leq n\}$ be a δ-fine free tagged partition of $[0, 1]$. For each i, with $1 \leq i \leq n$, let $U_i = \left(-\frac{\epsilon}{2^{\alpha(t_i)}}, \frac{\epsilon}{2^{\alpha(t_i)}}\right)$. If $t_i \in [0, 1] \cap \mathbf{Q}^c$,

$$F(x_i) - F(x_{i-1}) - (x_i - x_{i-1})f(t_i) = 0 \in U_i.$$

Suppose $t_i \in [0, 1] \cap \mathbf{Q}$. Note that $[x_{i-1}, x_i] \subseteq (t_i - \delta(t_i), t_i + \delta(t_i))$. This implies that $x_i - x_{i-1} < 2\delta(t_i) = \frac{\epsilon}{2^{\alpha(t_i)}}$. Consequently,

$$F(x_i) - F(x_{i-1}) - (x_i - x_{i-1})f(t_i) = -(x_i - x_{i-1}) \in U_i.$$

We now show that $\sum_{i=1}^{n} U_i \subseteq U$. Let $x \in \sum_{i=1}^{n} U_i$. Then $x = \sum_{i=1}^{n} x_i$, where $x_i \in U_i$. Let $m = \max\{\alpha(t_i) : 1 \leq i \leq n\}$. Since α is bijective,

$$\sum_{i=1}^{n} \frac{\epsilon}{2^{\alpha(t_i)}} \leq \sum_{k=1}^{m} \frac{\epsilon}{2^k}.$$

In effect,

$$\sum_{i=1}^{n} \frac{\epsilon}{2^{\alpha(t_i)}} < \sum_{k=1}^{+\infty} \frac{\epsilon}{2^k} = \epsilon.$$

Since $-\frac{\epsilon}{2^{\alpha(t_i)}} < x_i < \frac{\epsilon}{2^{\alpha(t_i)}}$,

$$-\sum_{i=1}^{n} \frac{\epsilon}{2^{\alpha(t_i)}} < x < \sum_{i=1}^{n} \frac{\epsilon}{2^{\alpha(t_i)}}.$$

Thus, $-\epsilon < x < \epsilon$. Hence, $x \in (-\epsilon, \epsilon) \subseteq U$. Accordingly, $\sum_{i=1}^{n} U_i \subseteq U$.

Hence, by Theorem 2.9, f is McShane integrable with M-primitive F.

Let us consider the Banach-version of an M-integrable Banach-valued functions on $[a, b]$. Let $(X, \| \ \|)$ be a Banach space. A function $f : [a, b] \to X$ is **M-integrable** on $[a, b]$, and we write $f \in M([a, b], X)$, if there is an $\alpha \in X$ such that for any $\varepsilon > 0$ there is a positive function δ on $[a, b]$ such that whenever a δ-fine free tagged partition P of $[a, b]$ is given, we have $\|S(f; P) - \alpha\| < \varepsilon$.

In the next theorem we show that if X is a Banach space, the two definitions are equivalent.

Theorem 2.11. *Let $(X, \| \ \|)$ be a Banach space. Then, the Banach version and the TVS version are equivalent.*

Proof. First, we show that the Banach version implies the TVS version. Let U be a θ-nbd. Then there exists $\varepsilon > 0$ such that $B_\varepsilon \subseteq U$ where $B_\varepsilon = \{x \in X : \|x\| < \varepsilon\}$. Thus, there is a positive function δ on $[a, b]$ such that whenever a δ-fine free tagged partition P of $[a, b]$ is given, we have $\|S(f; P) - \alpha\| < \varepsilon$. Hence, there is a positive function δ on $[a, b]$ such that whenever a δ-fine free tagged partition P of $[a, b]$ is given, we have $S(f; P) - \alpha \in B_\varepsilon \subseteq U$.

Next, we show that the TVS version implies the Banach version. Let $\varepsilon > 0$. Then, there is a positive function δ on $[a, b]$ such that whenever a δ-fine free tagged partition P of $[a, b]$ is given, we have $S(f; P) - \alpha \in B_\varepsilon$. Thus, there is a positive function δ on $[a, b]$ such that whenever a δ-fine free tagged partition P of $[a, b]$ is given, we have $\|S(f; P) - \alpha\| < \varepsilon$. □

References

[1] R.A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, American Mathematical Society, 1994.

[2] P.Y. Lee, *Lanzhou Lectures on Henstock Integration*, World Scientific Publishing Co. Pte. Ltd., 1989.

[3] A. Mukherjea and K. Pothoven, *Real and Functional Analysis*, 2nd ed., Plenum Press, New york, 1986.

[4] A. Grothendieck, *Topological Vector Spaces*, Gordon and Breach Science Publishers, 1973.

[5] J.B. Conway, *A First Course in Functional Analysis*, 2nd ed., Springer Verlag, 1997.

[6] H.L. Royden, *Real Analysis*, 3rd ed., Prentice-Hall Inc., 1988.

[7] J. Dugundji, *Topology*, Prentice Hall, 1975.

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