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Functions and differences whose roots have the same real part

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Abstract: Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function of order ≤ 1. Assume that all the roots of f have the same real part, abbreviated by $f \in SRP$. For $\lambda \in \mathbb{R} \setminus \{0\}$, define $\Delta_\lambda f(z) := f(z+\lambda) - f(z)$. We investigate the situation when $\Delta_\lambda f \in SRP$.

Keywords: entire functions, differences, zeros, real part

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1 Introduction

For $\lambda \in \mathbb{R} \setminus \{0\}$, define the difference operator Δ_λ by

$$\Delta_\lambda f(z) := f(z + \lambda) - f(z) \qquad \text{where } f : \mathbb{C} \to \mathbb{C}.$$

Define the class of entire functions all of whose roots have the same real part by

$$SRP := \{\text{entire } f : \mathbb{C} \to \mathbb{C};\ \text{all the roots of } f \text{ have the same real part}\}.$$

In [1], it is shown that both the polynomials x^n and $\Delta_\lambda(x^n) \in$ SRP. There then arises a natural question whether there are any other functions with this property.

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In this paper, we show that the answer is affirmative for entire functions of order < 1, and holds with additional assumptions for entire functions of order 1.

Recall that an entire function is a function analytic throughout $\mathbb{C}$. Let $f(z)$ be an entire function. The *order* of $f(z)$ is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\ln\ln M(r;f)}{\ln r},$$

where $M(r;f) = \max_{|z|=r} |f(z)|$. Examples of entire functions of order 0 are polynomials, while $\cos(\sqrt{z})$ is entire of order $1/2$. Examples of entire functions of order 1 are e^z, $\sin z$ and $\cos z$.

Let $f(z)$ be an entire function of finite order ρ with $\{z_n\}$ being the set of all its non-vanishing zeros. The Weierstrass prime factor associated with f is defined as

$$E\left(\frac{z}{z_n}, p\right) = \left(1 - \frac{z}{z_n}\right)\exp\left(\frac{z}{z_n} + \frac{1}{2}\left(\frac{z}{z_n}\right)^2 + \frac{1}{3}\left(\frac{z}{z_n}\right)^3 \cdots + \frac{1}{p}\left(\frac{z}{z_n}\right)^p\right)$$

$(p \in \mathbb{N}\cup\{0\})$ and the canonical associated with f, which is also an entire function, is defined by

$$P(z) = \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p\right).$$

Our main tool is the well-known Hadamard's factorization theorem of entire functions, [2], [3], which states that:

$$f(z) = e^{g(z)} z^k P(z),$$

where $g(z) \in \mathbb{C}[z]$ is a polynomial of degree $q \le \rho$, $k \in \mathbb{N}\cup\{0\}$, and $P(z)$ is a canonical product associated with f of order $\sigma \le \rho$.

It is also known that

- if $\rho \notin \mathbb{Z}$, then $\sigma = \rho$ and $q \le [\rho]$;
- if $\rho \in \mathbb{Z}$, then at least one of the quantities q and σ equals ρ;
- if $\sigma \notin \mathbb{Z}$, then $p = [\sigma]$;
- if $\sigma \in \mathbb{Z}$, then $p = \sigma$ if $\sum_{n=1}^{\infty} 1/|z_n|^{\sigma} = \infty$ and $p = \sigma - 1$ if $\sum_{n=1}^{\infty} 1/|z_n|^{\sigma} < \infty$.

The nonnegative integer p is called the *genus* of the canonical product P.

2 Results

Our first main result shows that the SRP property propagates from f to $\Delta_\lambda f$ for entire functions of order < 1. This generalizes the result of Evans, Stolarsky and Wavik in [1].

Theorem 2.1. *Let the notation be as set out in Section 1. If $f \in SRP$ and order$(f) := \rho < 1$, then $\Delta_\lambda f \in SRP$.*

Proof. Since $\rho < 1$, we have $q = 0$, $\sigma = \rho < 1$ and $p = 0$ and Hadamard's theorem yields

$$f(z) = e^{g_0} z^k \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \qquad (g_0 \in \mathbb{C}).$$

Since $f \in$ SRP, we can write the non-vanishing zeros of f as

$$z_n = c + d_n i \quad (c, d_n \in \mathbb{R}).$$

Take a root $w = a + bi \quad (a, b \in \mathbb{R})$ of $\Delta_\lambda f$. We now consider two cases depending on whether $k = 0$.

Case 1: $k = 0$. We aim to show that $\Re(w) = c - \frac{\lambda}{2}$. Since

$$\Delta_\lambda f(z) = e^{g_0} \left\{ \prod_{n=1}^{\infty} \left(1 - \frac{z + \lambda}{z_n}\right) - \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \right\},$$

each root $w = a + bi$ of $\Delta_\lambda f(z)$ must satisfy $\prod_{n=1}^{\infty} \left(1 - \frac{w+\lambda}{z_n}\right) = \prod_{n=1}^{\infty} \left(1 - \frac{w}{z_n}\right)$, i.e.,

$$\prod_{n=1}^{\infty} \{(c - a - \lambda) + (d_n - b)i\} = \prod_{n=1}^{\infty} \{(c - a) + (d_n - b)i\}.$$

Taking the absolute values of both sides, we get

$$\prod_{n=1}^{\infty} \left\{(c - a - \lambda)^2 + (d_n - b)^2\right\} = \prod_{n=1}^{\infty} \left\{(c - a)^2 + (d_n - b)^2\right\},$$

which necessarily imples that

$$(c - a - \lambda)^2 = (c - a)^2.$$

Thus, $\Re(w) = a = c - \frac{\lambda}{2}$, as desired.

Case 2: $k > 0$. We aim to show that $\Re(w) = -\frac{\lambda}{2}$. Since $k > 0$, the function f has 0 as one of its roots and as $f \in$ SRP, all its roots have real parts $= 0$. From

$$\Delta_\lambda f(z) = e^{g_0}\left\{(z+\lambda)^k \prod_{n=1}^{\infty}\left(1 - \frac{z+\lambda}{z_n}\right) - z^k \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right)\right\},$$

we see that each root $w = a + bi$ of $\Delta_\lambda f(z)$ must satisfy

$$(a+\lambda+bi)^k \prod_{n=1}^{\infty}\{(-a-\lambda)+(d_n-b)i\} = (a+bi)^k \prod_{n=1}^{\infty}\{-a+(d_n-b)i\}.$$

Taking the absolute values of both sides, we get

$$\left\{(a+\lambda)^2+b^2\right\}^k \prod_{n=1}^{\infty}\left\{(a+\lambda)^2+(d_n-b)^2\right\} = (a^2+b^2)^k \prod_{n=1}^{\infty}\left\{a^2+(d_n-b)^2\right\},$$

which forces $a^2 = (a+\lambda)^2$. Thus, $\Re(w) = a = -\frac{\lambda}{2}$.

Since w is an arbitrary root of $\Delta_\lambda f$, we conclude at once from both cases that $\Delta_\lambda f \in$ SRP. $\square$

We come now to our second main result.

Theorem 2.2. *Let the notation be as set out in Section 1. Assume that order* $(f) = \rho = 1$ *and* $p = 0$ *so that by Hadamard's theorem*

$$f(z) = e^{hz+g_0} z^k \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right),$$

where $h, g_0 \in \mathbb{C}$. *If* $f \in$ *SRP and* $\left|e^{h\lambda}\right| = 1$, *then* $\Delta_\lambda f \in$ *SRP.*

Proof. Again we treat two separate cases depending on whether $k = 0$. Since $f \in$ SRP, we can write the non-vanishing zeros of f as

$$z_n = c + d_n i \quad (c, d_n \in \mathbb{R}).$$

Take a root $w = a + bi \quad (a, b \in \mathbb{R})$ of $\Delta_\lambda f$.

Case 1: $k = 0$. We aim to show that $\Re(w) = c - \frac{\lambda}{2}$. Since

$$\Delta_\lambda f(z) = e^{h(z+\lambda)+g_0} \prod_{n=1}^{\infty}\left(1 - \frac{z+\lambda}{z_n}\right) - e^{hz+g_0} \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right),$$

each root $w = a + ib$ of $\Delta_\lambda f(z)$ satisfies

$$e^{h\lambda} \prod_{n=1}^{\infty} \{(c - a - \lambda) + (d_n - b)i\} = \prod_{n=1}^{\infty} \{(c - a) + (d_n - b)i\}.$$

Taking absolute values of both sides, we get

$$\left|e^{h\lambda}\right|^2 \prod_{n=1}^{\infty} \left\{(c - a - \lambda)^2 + (d_n - b)^2\right\} = \prod_{n=1}^{\infty} \left\{(c - a)^2 + (d_n - b)^2\right\}.$$

Using $\left|e^{h\lambda}\right| = 1$, we deduce at once that $(c - a - \lambda)^2 = (c - a)^2$,i.e., $\Re(w) = a = c - \frac{\lambda}{2}$.

Case 2: $k > 0$. We aim to show that $\Re(w) = -\frac{\lambda}{2}$. Since $k > 0$, the function f has 0 as one of its roots and as $f \in$ SRP, we infer that all its roots have real parts $= 0$. Since

$$\Delta_\lambda f(z) = e^{h(z+\lambda)+g_0}(z + \lambda)^k \prod_{n=1}^{\infty}(1 - \frac{z + \lambda}{z_n}) - e^{hz+g_0} z^k \prod_{n=1}^{\infty}(1 - \frac{z}{z_n}),$$

each root $w = a + ib$ of $\Delta_\lambda f(z)$ satisfies

$$e^{h\lambda}(a + \lambda + bi)^k \prod_{n=1}^{\infty} \{(-a - \lambda) + (d_n - b)i\} = (a + bi)^k \prod_{n=1}^{\infty} \{-a + (d_n - b)i\},$$

which yields after taking absolute on both sides

$$\left|e^{h\lambda}\right|^2 \left\{(a + \lambda)^2 + b^2\right\}^k \prod_{n=1}^{\infty} \left\{(a + \lambda)^2 + (d_n - b)^2\right\} = (a^2+b^2)^k \prod_{n=1}^{\infty} \left\{a^2 + (d_n - b)^2\right\}.$$

Using $\left|e^{h\lambda}\right| = 1$, we deduce that $(a + \lambda)^2 = a^2$, and so $\Re(w) = a = -\frac{\lambda}{2}$.

We conclude from both cases that $\Delta_\lambda f(z) \in$ SRP. □

Theorem 2.3. *Let the notation be as set out in Section 1. Assume that order $(f) = \rho = 1$ and $p = 1$ so that by Hadamard's theorem*

$$f(z) = e^{hz+g_0} z^k \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n}\right),$$

where $h, g_0 \in \mathbb{C}$.

(i) If $f \in SRP$, $k = 0$, $\left|e^{h\lambda}\right| = 1$ and $\left|\exp(\frac{\lambda}{z_n})\right| = 1$ (i.e., all non-vanishing zeros are purely imaginary), then $\Delta_\lambda f \in SRP$.

(ii) If $f \in$ SRP, $k > 0$ and $\left|e^{h\lambda}\right| = 1$, then $\Delta_\lambda f \in SRP$.

Proof. Again we treat two separate cases depending on whether $k = 0$. Since $f \in$ SRP, we can write the non-vanishing zeros of f as

$$z_n = c + d_n i \quad (c, d_n \in \mathbb{R}).$$

Take a root $w = a + bi \quad (a, b \in \mathbb{R})$ of $\Delta_\lambda f$.

Case 1: $k = 0$. We aim to show that $\Re(w) = c - \frac{\lambda}{2}$. Since

$$\Delta_\lambda f(z) = e^{h(z+\lambda)+g_0} \prod_{n=1}^{\infty} \left(1 - \frac{(z+\lambda)}{z_n}\right) \exp\left(\frac{(z+\lambda)}{z_n}\right) - e^{hz+g_0} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n}\right),$$

each root $w = a + ib$ of $\Delta_\lambda f(z)$ satisfies

$$e^{h\lambda} \prod_{n=1}^{\infty} \{(c - a - \lambda) + (d_n - b)i\} \exp\left(\frac{\lambda}{z_n}\right) = \prod_{n=1}^{\infty} \{(c - a) + (d_n - b)i\}.$$

Taking absolute values of both sides, we get

$$\left|e^{h\lambda}\right|^2 \prod_{n=1}^{\infty} \left\{(c - a - \lambda)^2 + (d_n - b)^2\right\} \left|\exp\left(\frac{\lambda}{z_n}\right)\right|^2 = \prod_{n=1}^{\infty} \left\{(c - a)^2 + (d_n - b)^2\right\}.$$

Using $\left|e^{h\lambda}\right| = 1$ and $\left|\exp(\frac{\lambda}{z_n})\right| = 1$, we deduce at once that $(c - a - \lambda)^2 = (c - a)^2$,i.e., $\Re(w) = a = c - \frac{\lambda}{2}$. **Case 2:** $k > 0$. We aim to show that $\Re(w) = -\frac{\lambda}{2}$. Since $k > 0$, the function f has 0 as one of its roots and as $f \in$ SRP, we infer that all its roots have real parts $= 0$. Since

$$\Delta_\lambda f(z) = e^{h(z+\lambda)+g_0} (z + \lambda)^k \prod_{n=1}^{\infty} \left(1 - \frac{(z+\lambda)}{z_n}\right) \exp\left(\frac{(z+\lambda)}{z_n}\right) - e^{hz+g_0} z^k \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n}\right),$$

each root $w = a + ib$ of $\Delta_\lambda f(z)$ satisfies

$$e^{h\lambda}(a+\lambda+bi)^k \prod_{n=1}^{\infty} \{(-a - \lambda) + (d_n - b)i\} \exp\left(\frac{\lambda}{z_n}\right) = (a+bi)^k \prod_{n=1}^{\infty} \{-a + (d_n - b)i\},$$

which yields after taking absolute on both sides

$$\left|e^{h\lambda}\right|^2 \left\{(a + \lambda)^2 + b^2\right\}^k \prod_{n=1}^{\infty} \left\{(a + \lambda)^2 + (d_n - b)^2\right\} \left|\exp\left(\frac{\lambda}{z_n}\right)\right|^2 = (a^2 + b^2)^k \prod_{n=1}^{\infty} \left\{a^2 + (d_n - b)^2\right\}.$$

Since $z_n = d_n i$, we have $\left|\exp\left(\frac{\lambda}{z_n}\right)\right| = 1$. Using $\left|e^{h\lambda}\right| = 1$, we deduce that $(a+\lambda)^2 = a^2$, and so $\Re(w) = a = -\frac{\lambda}{2}$.

We conclude from both cases that $\Delta_\lambda f(z) \in$ SRP. □

Theorem 2.3 can be stated in a slightly shorter version, by combining the two possibilities, as: Assume that order $(f) = \rho = 1$ and $p = 1$ so that by Hadamard's theorem

$$f(z) = e^{hz+g_0} z^k \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n}\right),$$

where $h, g_0 \in \mathbb{C}$. If $f \in$ SRP, all non-vanishing zeros of f are purely imaginary and $\left|e^{h\lambda}\right| = 1$, then $\Delta_\lambda f \in SRP$.

3 Examples

In this section, we work out several examples starting with entire functions of order 0 which extend the class obtained in [1]. The notation in Section 1 is being kept here.

Example 3.1. (Functions of order < 1)
I. Polynomials of the form

$$f(z) = (z+t)^n \quad (t \in \mathbb{R},\ n \in \mathbb{N}),$$

have orders 0. The single root of $f(z)$ is at $z = -t$ so that trivially $f \in$ SRP. Since

$$\Delta_\lambda (z+t)^n = (z+\lambda+t)^n - (z+t)^n,$$

the roots w of $\Delta_\lambda (z+t)^n$ satisfy $(w+\lambda+t)^n = (w+t)^n$. It is easily checked that these roots are

$$w_k = -t - \frac{\lambda}{2} - i\frac{\lambda \sin\frac{2\pi k}{n}}{2\left(1 - \cos\frac{2\pi k}{n}\right)} \quad (k = 1, \ldots, n-1),$$

all of which clearly have the same real part $-t - \frac{\lambda}{2}$.

II. The logarithmic function

$$f(z) = \log\left(z^n + 1\right) \quad (n \in \mathbb{N})$$

has order 0. Its sole zero is at $z = 0$, so that this function belongs to SRP. The roots w of $\Delta_\lambda \log(z^2 + 1) = 0$, are

$$w_k = -\frac{\lambda}{2} - i\frac{\lambda \sin \frac{2\pi k}{n}}{2\left(1 - \cos \frac{2\pi k}{n}\right)} \quad (k = 1, \ldots, n-1),$$

all of which have the same real part $\Re(w) = -\frac{\lambda}{2}$.

The next batch of examples are entire functions of order 1.

Example 3.2. (Functions of order 1)
I. The cosine function

$$f(z) = \cos(iz) = \prod_{n=1}^{\infty}\left(1 + \frac{4z^2}{(2n-1)^2\pi^2}\right)$$

is entire of order 1 with genus $p = 0$. Here, $h = 0$ so that $\left|e^{h\lambda}\right| = 1$. The zeros of $\cos(iz)$ are

$$z_n = -i\frac{(2n-1)\pi}{2} \quad (n \in \mathbb{Z}),$$

and so their real parts are equal to $\Re(z_n) = 0$, showing that $\cos(iz) \in$ SRP. The roots w of

$$0 = \Delta_\lambda \cos(iz) = \cos\left(i(z + \lambda)\right) - cos(iz)$$

are

$$w_n = -\frac{\lambda}{2} - n\pi i \quad (n \in \mathbb{Z}),$$

all of which have the same real parts equal to $-\frac{\lambda}{2}$, and so $\Delta_\lambda \cos(iz) \in$ SRP.

II. The sine function

$$f(z) = \sin(iz) = iz\prod_{n=1}^{\infty}\left(1 + \frac{4z^2}{n^2\pi^2}\right)$$

is entire of order 1 with $p = 0$. Here, $h = 0$ so that $\left|e^{h\lambda}\right| = 1$. The same analysis as in the last example shows that both $\sin(iz)$ and $\Delta_\lambda \sin(iz)$ are in SRP.

III. The exponential function

$$f(z) = e^{iz}z$$

is entire of order 1 with only a single zero at $z = 0$, and so it belongs to SRP with $p = 0$. Here, $\left|e^{i\lambda}\right| = 1$. The roots w of $0 = \Delta_\lambda e^{iz}z = e^{i(z+\lambda)}(z+\lambda) - e^{iz}z$ are

$$w = -\frac{\lambda}{2} + i\frac{\lambda\sin\lambda}{2 - 2\cos\lambda},$$

all of whose real parts are equal to $-\frac{\lambda}{2}$, showing that $\Delta_\lambda e^{iz}z \in$ SRP.

IV. The exponential polynomial

$$f(z) = e^{iz}(z+i)(z-i) = e^{iz}\left(1 + \frac{z}{i}\right)\left(1 - \frac{z}{i}\right)$$

is entire of order 1 and has two roots $\pm i$ so that $p = 0$. Here, $\left|e^{i\lambda}\right| = 1$. Let $w = a + bi$ $(a, b \in \mathbb{R})$ be any root of $\Delta_\lambda f$. Solving $\Delta_\lambda e^{iw}(w+i)(w-i) = 0$, we get

$$e^{i\lambda}(w + \lambda + i)(w + \lambda - i) = (w+i)(w-i).$$

Taking absolute values of bothe sides leads to

$$\left((a+\lambda)^2 + (b+1)^2\right)\left((a+\lambda)^2 + (b-1)^2\right) = \left(a^2 + (b+1)^2\right)\left(a^2 + (b-1)^2\right),$$

we deduce that $(a+\lambda)^2 = a^2$, which in turns yields $\Re(w) = a = -\frac{\lambda}{2}$, i.e. $\Delta_\lambda e^{iz}(z+i)(z-i) \in$ SRP.

V. The gamma function

$$f(z) = \frac{1}{\Gamma(iz)} = ize^{i\gamma z}\prod_{n=1}^{\infty}\left(1 + \frac{zi}{n}\right)e^{-\frac{zi}{n}} \qquad (\gamma, \text{Euler's constant})$$

is entire of order 1, [3], p.90, and has roots at 0 and ni $(n \in \mathbb{N})$, so that $p = 1$. Here, $\left|e^{i\gamma\lambda}\right| = 1$ and $\left|e^{-\frac{\lambda i}{n}}\right| = 1$. Let $w = a + bi$ $(a, b \in \mathbb{R})$ be any root of $\Delta_\lambda f$. Solving $\Delta_\lambda iwe^{i\gamma w}\prod_{n=1}^{\infty}\left(1 + \frac{wi}{n}\right)e^{-\frac{wi}{n}} = 0$, we get

$$\left(\frac{iw + i\lambda}{iw}\right)e^{i\gamma\lambda}\prod_{n=1}^{\infty}\left(\frac{n + iw + i\lambda}{n + iw}\right)e^{-\frac{\lambda i}{n}} = 1.$$

Taking absolute values of both sides leads to

$$\left(\frac{(a+\lambda)^2 + b^2}{a^2 + b^2}\right)\prod_{n=1}^{\infty}\left(\frac{(n+b)^2 + (a+\lambda)^2}{(n+b)^2 + a^2}\right) = 1$$

which in turns yields $\Re(w) = a = -\frac{\lambda}{2}$, i.e., $\Delta_\lambda \frac{1}{\Gamma(iz)} \in$ SRP.

The next example shows that the condition on the absolute value of the exponential factor cannot be dropped in general.

Example 3.3. The exponential polynomial

$$f(z) = e^z z^2$$

is entire of order 1 and has a single root at 0 with multiplicity 2, so that $p = 0$. Here, $\left|e^{\lambda}\right| \neq 1$. The roots w of $0 = \Delta_\lambda e^z z^2 = e^{z+\lambda}(z+\lambda)^2 - e^z z^2$ are

$$w = \frac{\lambda}{1 \pm e^{-\frac{\lambda}{2}}} \ .$$

showing,that $\Delta_\lambda e^{iz} z \notin$ SRP.

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