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Certain Maximal Commutative Subrings of Full Matrix Rings

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Abstract: Denote by $M_n(R)$ the full matrix ring over a commutative ring R with identity where $n > 1$. In this paper, we show that the set $D_n(R)$ of all matrices in $M_n(R)$ of the form

$$\begin{bmatrix} x_1 & 0 & \cdots & 0 & y_1 \\ 0 & x_2 & \cdots & y_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & y_2 & \cdots & x_2 & 0 \\ y_1 & 0 & \cdots & 0 & x_1 \end{bmatrix}$$

is a maximal commutative subring of the ring $M_n(R)$.

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1 Introduction

A *maximal commutative subring* of a ring R is defined naturally to be a maximal element of the set of all proper commutative subrings of R under inclusion. If R is a noncommutative ring, then a maximal commutative subring of R is a maximal element of the set of all commutative subrings of R under inclusion. The following proposition is clearly seen.

Proposition 1.1. *If S is a commutative subring of a noncommutative ring R such that for $x \in R$, $xa = ax$ for all $a \in S$ implies $x \in S$, then S is a maximal commutative subring of R.*

Throughtout, let R be a commutative ring with identity $1 \neq 0$ and n a positive integer greater than 1. Denote by $M_n(R)$ the full $n \times n$ matrix ring over R. Since $n > 1$, we have that $M_n(R)$ is a noncommutative ring. For $A \in M_n(R)$ and $i, j \in \{1, \ldots, n\}$, let A_{ij} denote the entry of A in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column.

Example 1.2. Let T be the set of all diagonal matrices of $M_n(R)$, that is,

$$T = \left\{ \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & a_n \end{bmatrix} \;\middle|\; a_1, \ldots, a_n \in R \right\}.$$

Then T is a maximal commutative subring of the ring $M_n(R)$. To see this, let $A \in M_n(R)$ be such that $AB = BA$ for all $B \in T$. Let $k, l \in \{1, \ldots, n\}$ be distinct. Define $E \in T$ by

$$E_{ij} = \begin{cases} 1 & \text{if } i = j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Then $AE = EA$. But $(AE)_{kl} = \sum_{i=1}^{n} A_{ki}E_{il} = 0$ and $(EA)_{kl} = \sum_{i=1}^{n} E_{ki}A_{il} = A_{kl}$, so we have that $A_{kl} = 0$. This shows that $A \in T$. By Proposition 1.1, T is a maximal commutative subring of the ring $M_n(R)$.

Kim Jin Bai [1] introduced some maximal commutative subsemigroups of the multiplicative semigroup of all $n\times n$ matrices over the semiring $([0,1],\max,\min)$. In [2], the authors proved that the sets $U_n(F)$ and $L_n(F)$ consisting of all $A\in M_n(F)$ of the forms

$$\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ 0 & x_1 & x_2 & \cdots & x_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & x_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1 & 0 & 0 & \cdots & 0 \\ x_2 & x_1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n & x_{n-1} & x_{n-2} & \cdots & x_1 \end{bmatrix},$$

respectively, are maximal commutative subrings of the ring $M_n(F)$ where F is a field. In fact, their proofs show that these results hold for the ring $M_n(R)$.

In this paper, we shall show that the set $D_n(R)$ consisting of all $A\in M_n(R)$ of the form

$$\begin{bmatrix} x_1 & 0 & \cdots & 0 & y_1 \\ 0 & x_2 & \cdots & y_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & y_2 & \cdots & x_2 & 0 \\ y_1 & 0 & \cdots & 0 & x_1 \end{bmatrix}$$

is a maximal commutative subring of the ring $M_n(R)$. This means that if n is even, then $D_n(R)$ is the set of all $A\in M_n(R)$ of the form

$$\begin{bmatrix} x_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & y_1 \\ 0 & \ddots & \ddots & & & \iddots & \iddots & 0 \\ & & & 0 & 0 & & & \\ \vdots & & 0 & x_m & y_m & 0 & & \vdots \\ \vdots & & 0 & y_m & x_m & 0 & & \vdots \\ & & & 0 & 0 & & & \\ 0 & \iddots & \iddots & & & \ddots & \ddots & 0 \\ y_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_1 \end{bmatrix} \quad \text{where} \quad n=2m$$

and if n is odd, then $D_n(R)$ is the set of all $A\in M_n(R)$ of the form

$$\begin{bmatrix} x_1 & 0 & \cdots & \cdots & \cdots & 0 & y_1 \\ 0 & \ddots & \ddots & & \iddots & \iddots & 0 \\ \vdots & \ddots & x_m & 0 & y_m & \iddots & \vdots \\ \vdots & & 0 & z & 0 & & \vdots \\ \vdots & \iddots & y_m & 0 & x_m & \ddots & \vdots \\ 0 & \iddots & \iddots & & \ddots & \ddots & 0 \\ y_1 & 0 & \cdots & \cdots & \cdots & 0 & x_1 \end{bmatrix} \quad \text{where} \quad n = 2m+1.$$

2 The Subring $D_n(R)$ of $M_n(R)$

First, we note that $aI_n \in D_n(R)$ for all $a \in R$ where I_n is the identity $n \times n$ matrix over R. Let

$$\Lambda = \begin{cases} \{1, \dots, \frac{n}{2}\} & \text{if } n \text{ is even,} \\ \{1, \dots, \frac{n-1}{2}\} & \text{if } n \text{ is odd.} \end{cases}$$

Then the following lemma is evident.

Lemma 2.1. *For $A \in M_n(R)$, $A \in D_n(R)$ if and only if*

(i) $A_{ii} = A_{n-i+1,n-i+1}$ *and* $A_{i,n-i+1} = A_{n-i+1,i}$ *for all* $i \in \Lambda$ *and*

(ii) $A_{ij} = 0$ *for all* $i, j \in \{1, \dots, n\}$ *with* $j \neq i$ *and* $j \neq n-i+1$.

To show that $D_n(R)$ is a maximal commutative subring of the ring $M_n(R)$, we first show that it is a commutative subring of $M_n(R)$.

Lemma 2.2. *The set $D_n(R)$ is a commutative subring of the ring $M_n(R)$.*

Proof. It is clearly seen that $D_n(R)$ is a subgroup of the group $(M_n(R), +)$. Let $A, B \in D_n(R)$. Then by Lemma 2.1,

$$\begin{aligned} A_{ii} &= A_{n-i+1,n-i+1}, \ A_{i,n-i+1} = A_{n-i+1,i}, \\ B_{ii} &= B_{n-i+1,n-i+1}, \ B_{i,n-i+1} = B_{n-i+1,i}, \\ & \qquad \text{for all } i \in \Lambda, \end{aligned} \tag{1}$$

$$A_{ij} = 0 = B_{ij} \text{ for all } i, j \in \{1, \ldots, n\} \text{ with } j \neq i \text{ and } j \neq n - i + 1. \qquad (2)$$

Note that $n - (n - i + 1) + 1 = i$ for all $i \in \{1, \ldots, n\}$. From (1) and (2), we have the following equalities for $i \in \Lambda$:

$$\begin{aligned}
(AB)_{ii} &= \sum_{k=1}^{n} A_{ik} B_{ki} \\
&= A_{ii} B_{ii} + A_{i,n-i+1} B_{n-i+1,i} \\
&= A_{n-i+1,n-i+1} B_{n-i+1,n-i+1} + A_{n-i+1,i} B_{i,n-i+1} \\
&= \sum_{k=1}^{n} A_{n-i+1,k} B_{k,n-i+1} \\
&= (AB)_{n-i+1,n-i+1} \,,
\end{aligned}$$

$$\begin{aligned}
(AB)_{ii} &= A_{ii} B_{ii} + A_{i,n-i+1} B_{n-i+1,i} \\
&= B_{ii} A_{ii} + B_{n-i+1,i} A_{i,n-i+1} \\
&= B_{ii} A_{ii} + B_{i,n-i+1} A_{n-i+1,i} \\
&= \sum_{k=1}^{n} B_{ik} A_{ki} \\
&= (BA)_{ii} \,,
\end{aligned}$$

$$\begin{aligned}
(AB)_{i,n-i+1} &= \sum_{k=1}^{n} A_{ik} B_{k,n-i+1} \\
&= A_{ii} B_{i,n-i+1} + A_{i,n-i+1} B_{n-i+1,n-i+1} \\
&= A_{n-i+1,n-i+1} B_{n-i+1,i} + A_{n-i+1,i} B_{ii} \\
&= \sum_{k=1}^{n} A_{n-i+1,k} B_{ki} \\
&= (AB)_{n-i+1,i} \,,
\end{aligned}$$

$$\begin{aligned}(AB)_{i,n-i+1} &= A_{ii}B_{i,n-i+1} + A_{i,n-i+1}B_{n-i+1,n-i+1} \\ &= B_{i,n-i+1}A_{n-i+1,n-i+1} + B_{ii}A_{i,n-i+1} \\ &= B_{ii}A_{i,n-i+1} + B_{i,n-i+1}A_{n-i+1,n-i+1} \\ &= \sum_{k=1}^{n} B_{ik}A_{k,n-i+1} \\ &= (BA)_{i,n-i+1}\,.\end{aligned}$$

Also, if $i, j \in \{1, \ldots, n\}$ are such that $j \neq i$ and $j \neq n - i + 1$, then from (2), we have

$$\begin{aligned}(AB)_{ij} &= \sum_{k=1}^{n} A_{ik}B_{kj} \\ &= A_{ii}B_{ij} + A_{i,n-i+1}B_{n-i+1,j} \\ &= A_{ii}0 + A_{i,n-i+1}0 = 0\end{aligned}$$

and

$$\begin{aligned}(BA)_{ij} &= \sum_{k=1}^{n} B_{ik}A_{kj} \\ &= B_{ii}A_{ij} + B_{i,n-i+1}A_{n-i+1,j} \\ &= B_{ii}0 + B_{i,n-i+1}0 = 0.\end{aligned}$$

Then $AB = BA$ and it follows from Lemma 2.1 that $AB \in D_n(R)$, so the desired result follows. □

Theorem 2.3. *The set $D_n(R)$ is a maximal commutative subring of the ring $M_n(R)$.*

Proof. It follows from Lemma 2.2 that $D_n(R)$ is a commutative subring of $M_n(R)$. To show the maximality of $D_n(R)$ by Proposition 1.1, let $A \in M_n(R)$ be such that

$$AX = XA \quad \text{for all } X \in D_n(R). \tag{1}$$

For each $l \in \Lambda$, let $E^{(l)} \in D_n(R)$ be defined by

$$E^{(l)}_{ij} = \begin{cases} 1 & \text{if } i = j = l \text{ or } i = j = n - l + 1, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

that is,

$$E^{(l)} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & & & & \\ & \ddots & 1 & \ddots & & & & & \vdots \\ \vdots & & \ddots & 0 & \ddots & & & & \\ & & & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & & \ddots & 1 & \ddots & & \\ & & & & & \ddots & 0 & \ddots & \vdots \\ \vdots & & & & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix} . \quad \begin{matrix} \leftarrow l^{\,\underline{th}} \text{ row} \\ \\ \leftarrow n-l+1^{\,\underline{th}} \text{ row} \end{matrix}$$

By (1), $AE^{(l)} = E^{(l)}A$ for all $l \in \Lambda$. If $l \in \Lambda$ and $j \in \{1, \ldots, n\}$ are such that $j \neq l$ and $j \neq n-l+1$, then from (2), we have

$$\begin{aligned} (AE^{(l)})_{lj} &= \sum_{k=1}^{n} A_{lk}E_{kj}^{(l)} = 0, \\ (E^{(l)}A)_{lj} &= \sum_{k=1}^{n} E_{lk}^{(l)}A_{kj} \\ &= E_{ll}A_{lj} = A_{lj}, \\ (AE^{(l)})_{n-l+1,j} &= \sum_{k=1}^{n} A_{n-l+1,k}E_{kj}^{(l)} = 0, \\ (E^{(l)}A)_{n-l+1,j} &= \sum_{k=1}^{n} E_{n-l+1,k}^{(l)}A_{kj} \\ &= E_{n-l+1,n-l+1}^{(l)}A_{n-l+1,j} = A_{n-l+1,j} \,. \end{aligned}$$

This proves that $A_{lj} = 0 = A_{n-l+1,j}$ for all $l \in \Lambda$ and $j \in \{1, \ldots, n\}$ with $j \neq l$ and $j \neq n-l+1$. Now, we have

$$A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 & A_{1n} \\ 0 & A_{22} & \cdots & A_{2,n-1} & 0 \\ \cdots & \cdots & \cdots & \cdots & \ldots \\ 0 & A_{n-1,2} & \cdots & A_{n-1,n-1} & 0 \\ A_{n1} & 0 & \cdots & 0 & A_{nn} \end{bmatrix} .$$

Then A satisfies (ii) of Lemma 2.1. Define $B \in D_n(R)$ by

$$B_{ij} = \begin{cases} 1 & \text{if } j = n-i+1, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

that is,

$$B = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & \iddots & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & \iddots & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

By (1), $AB = BA$. Then we have from (3) that for $i \in \Lambda$,

$$\begin{aligned}
(AB)_{i,n-i+1} &= \sum_{k=1}^{n} A_{ik}B_{k,n-i+1} = A_{ii}B_{i,n-i+1} = A_{ii}\,, \\
(BA)_{i,n-i+1} &= \sum_{k=1}^{n} B_{ik}A_{k,n-i+1} = B_{i,n-i+1}A_{n-i+1,n-i+1} = A_{n-i+1,n-i+1}\,, \\
(AB)_{ii} &= \sum_{k=1}^{n} A_{ik}B_{ki} = A_{i,n-i+1}B_{n-i+1,i} = A_{i,n-i+1}\,, \\
(BA)_{ii} &= \sum_{k=1}^{n} B_{ik}A_{ki} = B_{i,n-i+1}A_{n-i+1,i} = A_{n-i+1,i}\,,
\end{aligned}$$

which imply that $A_{ii} = A_{n-i+1,n-i+1}$ and $A_{i,n-i+1} = A_{n-i+1,i}$. This shows that A satisfies (i) of Lemma 2.1. It then follows from Lemma 2.1 that $A \in D_n(R)$. Hence by Proposition 1.1, $D_n(R)$ is a maximal commutative subring of the ring $M_n(R)$, as desired □

Remark 2.4. Let F be a field. The following properties of $D_n(F)$ are clearly seen.

(1) If F is a finite field of order q, then $|M_n(F)| = q^{n^2}$ while $|D_n(F)| = q^n$ where for a set X, $|X|$ stands for the cardinality of X.

(2) As vector spaces over F, $\dim M_n(F) = n^2$, $D_n(F)$ is a subspace of $M_n(F)$ and $\dim D_n(F) = n$. For $i \in \Lambda$, let

$$
B^{(i)} = \begin{bmatrix}
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & & & & & & \\
 & \ddots & 1 & \ddots & & & & & \vdots \\
\vdots & & \ddots & 0 & \ddots & & & & \\
 & & & \ddots & \ddots & \ddots & & & \vdots \\
\vdots & & & & \ddots & 1 & \ddots & & \\
 & & & & & \ddots & 0 & \ddots & \vdots \\
\vdots & & & & & & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0
\end{bmatrix},
\begin{matrix}
 \\
 \\
\leftarrow i^{\underline{th}} \ \text{row} \\
 \\
 \\
\leftarrow n-i+1^{\underline{th}} \ \text{row} \\
 \\
 \\
 \\
\end{matrix}
$$

$$
C^{(i)} = \begin{bmatrix}
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
 & & & & & & \iddots & \iddots & 0 \\
\vdots & & & & & \iddots & 0 & \iddots & \\
 & & & & \iddots & 1 & \iddots & & \vdots \\
\vdots & & & \iddots & \iddots & \iddots & & & \\
 & & \iddots & 0 & \iddots & & & & \vdots \\
\vdots & \iddots & 1 & \iddots & & & & & \\
0 & \iddots & \iddots & & & & & & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{bmatrix}.
\begin{matrix}
 \\
 \\
 \\
\leftarrow i^{\underline{th}} \ \text{row} \\
 \\
 \\
\leftarrow n-i+1^{\underline{th}} \ \text{row} \\
 \\
 \\
\end{matrix}
$$

If n is odd, let $K \in M_n(F)$ be as follows:

$$
K = \begin{bmatrix}
0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & & \vdots & & \iddots & \vdots \\
 & & 0 & 0 & 0 & & \\
\vdots & & 0 & 1 & 0 & & \vdots \\
 & & 0 & 0 & 0 & & \\
\vdots & \iddots & & \vdots & & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & \cdots & \cdots & 0
\end{bmatrix}.
$$

It is clear that if n is even, then $\{B^{(1)}, \ldots, B^{(\frac{n}{2})}, C^{(1)}, \ldots, C^{(\frac{n}{2})}\}$ is a basis of $D_n(F)$ over F and if n is odd, then $\{B^{(1)}, \ldots, B^{(\frac{n-1}{2})}, C^{(1)}, \ldots, C^{(\frac{n-1}{2})}, K\}$ is a basis of $D_n(F)$. Observe that for $A \in D_n(F)$,

$$A = A_{11}B^{(1)} + \cdots + A_{\frac{n}{2},\frac{n}{2}}B^{(\frac{n}{2})} + A_{1n}C^{(1)} + \cdots + A_{\frac{n}{2},\frac{n}{2}+1}C^{(\frac{n}{2})} \quad \text{if } n \text{ is even,}$$

$$A = A_{11}B^{(1)} + \cdots + A_{\frac{n-1}{2},\frac{n-1}{2}}B^{(\frac{n-1}{2})} + A_{1n}C^{(1)} + \cdots + A_{\frac{n-1}{2},\frac{n-1}{2}+2}C^{(\frac{n-1}{2})} + A_{\frac{n+1}{2},\frac{n+1}{2}}K \quad \text{if } n \text{ is odd.}$$

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