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Characterizations of a $*$-Almost Distributive Lattice

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Abstract: The space of minimal prime ideals of an almost distributive lattice (ADL) with zero, endowed with the hull kernel topology is shown to be a zero-dimensional, totally disconnected Tychonoff space. Several characterizations of a $*$-ADL using the topological properties of the space of minimal prime ideals are obtained. Necessary and sufficient conditions for the space of minimal prime ideals of an ADL and the space of minimal prime ideals of its distributive lattice of all ideals, to be homeomorphic are furnished.

Keywords: Almost Distributive Lattice (ADL), ideal, filter, minimal prime ideal, maximal filter, $*$-ADL, hull kernel topology

2000 Mathematics Subject Classification: 06D99

1 Introduction

Henrikisan and Jerison [2] investigated the space of minimal prime ideals of a commutative ring extending the consideration of Kist [3] in the context of commutative semigroup with 0. They succeeded in obtaining a sufficient condition for their respective spaces to be compact. This work inspired Speed [8, 9] to investigate minimal prime ideals of a distributive lattice with 0. Fortunately the nature of lattice theoretic situation enabled Speed [8] to obtain much deeper results and he

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could characterized the compactness of the space of minimal prime ideals of a distributive lattice with 0 in a much more elegant manner. With an idea of bringing a common abstraction to most of the existing ring theoretic and lattice theoretic generalization of Boolean algebra, the concept of an Almost Distributive Lattice (ADL) was introduced by Swamy and Rao in [10]. Hence it is worth studying the properties of minimal prime ideals and the space of minimal prime ideals in an ADL. All these considerations coupled with the properties of minimal prime ideals of an ADL obtained by Rao in [7] motivate us to carry out a detailed investigation of the space of minimal prime ideals of an ADL R with 0. The second section deals with basic concepts and results of an ADL. The concluding section is devoted to obtaining various characterizations of a $*$-ADL using topological properties of the space of minimal prime ideals. For the congruence relation θ defined on an ADL R with 0 by $x \equiv y(\theta)$ if and only if $\{x\}^* = \{y\}^* \, (x, y \in R)$, it is proved that the quotient ADL R/θ is a Boolean lattice if and only if the space of minimal prime ideals of R is compact. Necessary and sufficient conditions for the space of minimal prime ideals of an ADL R with 0 and the space of minimal prime ideals of $\mathcal{I}(R)$ to be homeomorphic are furnished where $\mathcal{I}(R)$ denotes the lattice of all ideals of R.

2 Preliminaries

At first we recall certain definitions and results mostly from [4], [5], [7] and [10] that we need in the sequel. An Almost Distributive Lattice (ADL) is an algebra $(R, \vee, \wedge, 0)$ of type $(2, 2, 0)$ satisfying the following axioms:

1. $a \vee 0 = a$,
2. $0 \wedge a = 0$,
3. $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$,
4. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
5. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
6. $(a \vee b) \wedge b = b, \qquad \forall\, a, b, c \in R$.

Throughout this paper, R stands for an ADL R with 0 unless otherwise mentioned. For any $a, b \in R$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently,

$a \vee b = b$), then $\leq$ is a partial ordering on R. A non empty subset I of R is said to be an ideal of R, if it satisfies the following conditions.

(i) $a, b \in I \Rightarrow a \vee b \in I$ and

(ii) $a \in I,\ x \in R \Rightarrow a \wedge x \in I$.

A proper ideal P of R is said to be prime if for any $x, y \in R,\ x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. A prime ideal P of R is called minimal if there exists no prime ideal Q of R such that $Q \subseteq P$. A proper ideal M of R is said to be maximal if it is not properly contained in any proper ideal of R. For a proper subset $S \subseteq R$ ideal generated by S in R is the smallest ideal of R containing S and is denoted by $(S]$. For $S = \{a\}$ it is simply denoted by $(a]$. For any non-empty subset A of an ADL R, define $A^* = \{x \in R \,|\, a \wedge x = 0, \text{for all}\, a \in A\}$. A^* is called the annihilator ideal of A. For $x \in R, (x]^*$ is called an annulet of R. An element $a \in R$ is called dense if $(a]^* = (0]$. Dually we can define filter, prime filter, minimal prime filter and maximal filter in R.

An equivalence relation θ on R is called a congruence relation if for all $a, b, c, d \in R,\ a \equiv b(\theta),\ c \equiv d(\theta) \Rightarrow a \wedge c \equiv b \wedge d(\theta),\ a \vee c \equiv b \vee d(\theta)$. For any congruence relation θ on R, we denote the congruence class containing $x \in R$ by $[x]^\theta$ and the set of all congruence classes of R is denoted by R/θ.

We need the following lemmas in the sequel.

Lemma 2.1. *[5] For any $a, b, c \in R$, we have the following:*

1. $a \vee b = a \Leftrightarrow a \wedge b = b$
2. $a \vee b = b \Leftrightarrow a \wedge b = a$
3. $\wedge$ *is associative in* R
4. $a \wedge b \wedge c = b \wedge a \wedge c$
5. $(a \vee b) \wedge c = (b \vee a) \wedge c$
6. $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
7. $a \wedge (b \vee a) = a \wedge (a \vee b) = a$
8. $a \leq a \vee b$ *and* $a \wedge b \leq b$
9. *If* $a \leq c, b \leq c$ *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$
10. $a \vee b = (a \vee b) \vee a$

Lemma 2.2. *[5] For an ideal I of R and $a, b \in R$, $a \wedge b \in I \Leftrightarrow b \wedge a \in I$.*

Lemma 2.3. *[5] The set $\mathcal{I}(R)$ of all ideals of R is a complete distributive lattice with the least element $\{0\}$ and the greatest element R in which for any $I, J \in \mathcal{I}(R), I \cap J = I \barwedge J$ is the infimum of I and J and the supremum is given by $I \veebar J = \{i \vee j \,|\, i \in I, j \in J\}$.*

Lemma 2.4. *[7] Let P be a prime ideal of an ADL R. Then P is minimal if and only if for each $x \in P$ there exist $y \notin P$ such that $x \wedge y = 0$.*

Lemma 2.5. *[7] Every prime ideal of R contains a minimal prime ideal.*

Lemma 2.6. *[7] A prime ideal P of an ADL R is minimal if and only if, for each $x \in P$ implies $(x]^* \nsubseteq P$.*

Lemma 2.7. *For any $0 \neq x \in R$, there exists a maximal filter F such that $x \in F$.*

Lemma 2.8. *[7] P is a minimal prime ideal if and only if $R \setminus P$ is a maximal filter.*

Lemma 2.9. *[6] The set D of all dense elements of R is a filter, provided $D \neq \emptyset$.*

Lemma 2.10. *[7] If I is an ideal and F a filter of R such that $I \cap F = \emptyset$, then there exists a prime filter G of R such that $F \subseteq G$ and $I \cap G = \emptyset$.*

Lemma 2.11. *[5] The set $\mathcal{A}_0(R)$ of all annulets of an ADL R forms a distributive lattice under the binary operations $\barwedge$ and $\veebar$ defined by $(x]^* \barwedge (y]^* = (x \vee y]^*$ and $(x]^* \veebar (y]^* = (x \wedge y]^*$ for any $(x]^*, (y]^* \in \mathcal{A}_0(R)$.*

Lemma 2.12. *[4] The set R/θ is an ADL, under the binary operations $\wedge$ and $\vee$ defined by $[x]^\theta \vee [y]^\theta = [x \vee y]^\theta$ and $[x]^\theta \wedge [y]^\theta = [x \wedge y]^\theta$ for all $[x]^\theta$ and $[y]^\theta \in R/\theta$.*

Lemma 2.13. *[11] R is a $*$-ADL if and only if $\mathfrak{M}(R)$ is a compact space.*

3 The space of minimal prime ideals

Let $\mathfrak{M}(R)$ be the set of all minimal prime ideals of R. For a subset $\mathcal{A}$ of $\mathfrak{M}(R)$, we write, as usual , the kernel of $\mathcal{A} = k(\mathcal{A}) = \bigcap\{B \,|\, B \in \mathcal{A}\}$ and for a subset $P \neq \emptyset$ of R the hull of $P = h(P) = \{M \in \mathfrak{M}(R) \,|\, P \subseteq M\}$. For $P = \{x\}$ we denote $h(\{x\})$ by $h(x)$ only. We may turn $\mathfrak{M}(R)$ into a topological space by endowing it with the so called hull kernel topology which has the sets of the form $V(x) = \{P \in \mathfrak{M}(R) \,|\, x \notin P\}(x \in R)$ as a base for the open sets. Obviously, $h(x) = \mathfrak{M}(R) \setminus V(x)$ and $V(x) = V(y)$ if and only if $\{x\}^{**} = \{y\}^{**}$ for all $x, y \in R$. We commence with some important properties of the hull and kernel in the space of minimal prime ideals $\mathfrak{M}(R)$ of R. These properties are crucial for characterizing a $*$- ADL. The proof of the next theorem can be gleaned from the proofs of analogous results for a distributive lattice given in [2] and hence is omitted.

Theorem 3.1. *Following properties hold in R:*

1. *For any ideal I of R, $I^* = k(\mathfrak{M}(R) \setminus h(I))$.*
2. *For any non-empty subset A of R with $A \neq \{0\}$, we have $A^* = k\left(h(A^*)\right)$.*
3. *A prime ideal M of R is minimal if and only if $x^* \setminus M \neq \emptyset$, for all $x \in M$.*

The following theorem is taken from [11].

Theorem 3.2. *Following properties hold in R:*

1. *$h\left(k(V(x))\right) = V(x) = h(\{x\}^*)$ for each $x \in R$.*
2. *$h(x) = h(\{x\}^{**})$ for each $x \in R$.*
3. *$z^* = x^* \cap y^*$ if and only if $h(z) = h(x) \cap h(y)$ for $x, y, z \in R$.*
4. *$\{x\}^{**} = \{y\}^*$ if and only if $h(x) = h(\{y\}^*)$ for $x, y \in R$.*

Remark 3.3. (a) $\mathfrak{M}(R)$ is a Hausdorff space (by Lemma 2.4)

(b) The space $\mathfrak{M}(R)$ is totally disconnected as base sets of the space $\mathfrak{M}(R)$ are open as well as closed (by Theorem 3.2).

(c) The space $\mathfrak{M}(R)$ is a zero dimensional space, as $\mathfrak{M}(R)$ is a totally disconnected, Hausdorff space .

Theorem 3.4. *The space $\mathfrak{M}(R)$ is completely regular and hence a Tychonoff space.*

Proof. Let M_1 be any element of $\mathfrak{M}(R)$ and $\mathcal{F}$ be any closed subset of $\mathfrak{M}(R)$ not containing the element M_1. Thus M_1 is in an open subset $\mathfrak{M}(R)\setminus\mathcal{F}$ of $\mathfrak{M}(R)$. This implies that there exists a neighborhood $V(x)$ of M_1 contained in $\mathfrak{M}(R)\setminus\mathcal{F}$, for some x in R. Define a function f on $\mathfrak{M}(R)$ as $f(M)=0$ for $M\in V(x)$ and $f(M)=1$ otherwise. We get $f(M_1)=0$ and $f(\mathcal{F}\setminus V(x))=1$. The continuity of f follows from from the fact that $V(x)$ is both open and closed (by Theorem 3.2). Thus the space $\mathfrak{M}(R)$ is completely regular. As $\mathfrak{M}(R)$ is a completely regular Hausdorff space it is a Tychonoff space. □

The next theorem deals with a property of an open set $V(I)$, where I is an ideal of R.

Theorem 3.5. *For any ideal I in R, $V(I)=\bigcup_{x\in I}V(x)=\bigcup_{x\in I}V((x]^{**})$.*

Proof. Let $P\in V(I)$. Select $x\in I$ such that $x\notin P$. As $x\notin P$ implies $(x]^{**}\nsubseteq P$. Hence $P\in V((x]^{**})$. This shows that $V(I)\subseteq\bigcup_{x\in I}V((x]^{**})$. Now if $P\in\bigcup_{x\in I}V((x]^{**})$, then $P\in V((x]^{**})$ for some $x\in I$. Thus $(x]^{**}\nsubseteq P$. As P is a minimal prime ideal, $x\notin P$. But then $I\nsubseteq P$ will imply $P\in V(I)$. This shows $\bigcup_{x\in I}V((x]^{**})\subseteq V(I)$. Hence $V(I)=\bigcup_{x\in I}V((x]^{**})$.
Let $P\in V(I)$. Then $I\nsubseteq P$ and $P\in\mathfrak{M}(R)$. Select $x\in I$ such that $x\notin P$. Then $P\in V(x)$ for $x\in I$. This in turn will imply $V(I)\subseteq\bigcup_{x\in I}V(x)$. Now if $P\in\bigcup_{x\in I}V(x)$, then $P\in V(x)$ for some $x\in I$. Then as $x\notin P$, we get $I\nsubseteq P$. Hence $P\in V(I)$. This shows $\bigcup_{x\in I}V(x)\subseteq V(I)$. Combining both the inclusions we get $V(I)=\bigcup_{x\in I}V(x)$. □

A property of the set $\{V(x)\,|\,x\in R\}$ is proved in the following theorem.

Theorem 3.6. *Let $\{x_r : r\in\Delta\}$ (Δ is any indexing set) be the set of elements in R such that the collection $\{V(x_r)\}$ has the finite intersection property. Then the intersection of all $\{V(x_r)\}$ is non-empty.*

Proof. Since any finite intersection $\bigcap_{r=1}^{n}V(x_r)$ is of the form $V(y)$ where $y=x_1\wedge x_2\wedge\cdots\wedge x_n$, it follows from the finite intersection property that the meet of any finite number of x_r, $r\in\Delta$ is non-zero. Hence the collection of all the elements x_r, $r\in\Delta$ together with their finite infima is a filter say F, in R. As $0\in R$, F is contained in some maximal filter, say Q, in R. Therefore $R\setminus Q$ is a minimal prime ideal by Lemma 2.8. Since no x_r is a member of $R\setminus Q$ and $R\setminus Q$ is a member of $\mathfrak{M}(R)$ we have $R\setminus Q\subseteq\bigcap_{r\in\Delta}V(x_r)$. Therefore $\bigcap_{r\in\Delta}V(x_r)\neq\emptyset$. □

4 Characterizations of a $*$-ADL

In this section we characterize $*$-ADL using the properties of the space of minimal prime ideals.

Recall that an ADL R with 0 is a $*$-ADL if for each $x \in R$, there exist $x' \in R$ such that $(x]^{**} = (x']^{*}$. An ADL R with 0 is a $*$-ADL if and only if to each $x \in R$, there exists $x' \in R$ such that $x \wedge x' = 0$ and $x \vee x'$ is dense (see [11]).

Example 4.1. Let X be any non-empty set. Fix $x_0 \in X$. For any $x, y \in X$, define binary operations $\vee$, $\wedge$ on X by

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \qquad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0 \end{cases}$$

Then $(X, \vee, \wedge, x_o)$ is an ADL which is called a discrete ADL with zero x_0. If $x \neq x_0$ is any element of R then $(x_0]^{*} = (x]^{**}$ and $(x]^{*} = (x_0]^{**}$, hence R is a $*$-ADL.

Example 4.2. Let $R = \{0, a, b, c\}$ and define $\wedge$ and $\vee$

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	b	b
b	b	b	b	b
c	c	b	b	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	a	b	c
c	0	0	c	c

Then R is an ADL. Also $(0]^{*} = (b]^{**}$, $(a]^{*} = (c]^{**}$, $(b]^{*} = (0]^{**}$, $(c]^{*} = (a]^{**}$, hence R is a $*$-ADL.

Theorem 4.3. *R is a $*$-ADL if and only if for each $x \in R$ there exists $y \in R$ such that $V(x) = h(y)$.*

Proof. Let R be a $*$-ADL. Fix up any $x \in R$ as R is $*$-ADL there exists $y \in R$ such that $\{x\}^{*} = \{y\}^{**}$. Now by Theorem 3.2, we have $V(x) = h(\{x\}^{*}) = h(\{y\}^{**}) = h(y)$.

Conversely, suppose for each $x \in R$ there exits $y \in R$ such that $V(x) = h(y)$. Fix up any $x \in R$. By assumption there exists $y \in R$ such that $V(x) = h(y)$. Therefore by Theorem 3.2, $h(\{x\}^{*}) = h(y)$ and hence $\{x\}^{*} = \{y\}^{**}$. This shows that R is a $*$-ADL. □

From Theorem 4.3 we have

Corollary 4.4. *R is a $*$-ADL if and only if* $\{V(x)\,|\,x \in R\} = \{h(y)\,|\,y \in R\}$.

Proof. By Theorem 4.3, $\{V(x)\,|\,x \in R\} = \{h(y)\,|\,y \in R\}$ implies R is a $*$-ADL. Conversely, let R be a $*$-ADL. Select $y \in R$. R being a $*$-ADL, there exists $x \in R$ such that $\{y\}^* = \{x\}^{**}$. Then by Theorem 3.2, $h(y) = h(\{y\}^{**}) = h(\{x\}^*) = V(x)$. This shows that $\{h(y)\,|\,y \in R\} \subseteq \{V(x)\,|\,x \in R\}$. But $\{V(x)\,|\,x \in R\} \subseteq \{h(y)\,|\,y \in R\}$ by Theorem 4.3. Combining both the inclusions we get $\{V(x)\,|\,x \in R\} = \{h(y)\,|\,y \in R\}$. □

One more characterization of a $*$-ADL in terms of the set $\{V(x)\,|\,x \in R\}$ is given in the following theorem.

Theorem 4.5. *R is a $*$-ADL if and only if* $< \{V(x)\,|\,x \in R\}, \cup, \cap >$ *is a Boolean lattice.*

Proof. Let R be a $*$-ADL. Then for any $x \in R$, there exists $y \in R$ such that $x \wedge y = 0$ and $x \vee y$ is a dense element. Therefore $V(x) \cap V(y) = \emptyset$ and $V(x) \cup V(y) = \mathfrak{M}(R)$. Hence $(V(x))' = V(y)$. This shows that the distributive lattice $< \{V(x)\,|\,x \in R\}, \cup, \cap >$ is a Boolean lattice.
Conversely, let $< \{V(x)\,|\,x \in R\}, \cup, \cap >$ is a Boolean lattice. Select any $x \in R$. Then for $V(x)$ there exists $y \in R$ such that $(V(x))' = V(y)$. Thus $V(x) \cap V(y) = \emptyset \Rightarrow V(x \wedge y) = \emptyset \Rightarrow x \wedge y = 0$. Also $V(x) \cup V(y) = \mathfrak{M}(R) \Rightarrow V(x \vee y) = \mathfrak{M}(R) \Rightarrow x \vee y$ is a dense element. Hence R is a $*$-ADL. □

Define a relation θ on R as $x \equiv y(\theta)$ if and only if $\{x\}^* = \{y\}^*$ for $x, y \in R$. Then θ is a congruence relation on R if R is a $*$-ADL. Using this fact we have

Theorem 4.6. *In a $*$-ADL R, R/θ is isomorphic with* $\{V(x)\,|\,x \in R\}$.

Proof. Define the map $f : \{V(x)\,|\,x \in R\} \rightarrow R/\theta$ as $f(V(x)) = [x]^\theta$ for each $x \in R$. If $V(x) = V(y)$ then $\{x\}^* = \{y\}^*$ (by Theorem 3.2). Hence $[x]^\theta = [y]^\theta$. This implies that f is well defined. Further $f(V(x) \cap V(y)) = f(V(x \wedge y)) = [x \wedge y]^\theta = [x]^\theta \cap [y]^\theta = f(V(x)) \cap f(V(y))$ it follows that f is a meet homomorphism. Similarly we can prove that f is a join homomorphism. Finally if $[x]^\theta = [y]^\theta$ then $\{x\}^* = \{y\}^*$ and this will imply that $V(x) = V(y)$. Hence f is an isomorphism. □

Let $\mathcal{I}(R)$ denote the set of all ideals in R. Then $<\mathcal{I}(R), \overline{\wedge}, \underline{\vee}>$ is a complete pseudo-complemented distributive lattice. Where $J\overline{\wedge}K = J\cap K$, $J\underline{\vee}K = (J\cup K]$ and $J^* = \{x \in R \,|\, x \wedge j = 0 \text{ for each}\, j \in J\}$ for $J, K \in \mathcal{I}(R)$. For a prime ideal $\mathcal{P}$ of $\mathcal{I}(R)$, $C(\mathcal{P})$ denotes the set theoretical union of all ideals (in R) which are in $\mathcal{P}$ i.e. $C(\mathcal{P}) = \cup\{J \in \mathcal{I}(R) \,|\, J \in \mathcal{P}\}$, while for a prime ideal Q in R, $\Gamma(Q)$ denotes the set $\{J \in \mathcal{I}(R) \,|\, J \subseteq Q\}$. It is easy to see that $C(\mathcal{P})$ and $\Gamma(Q)$ are prime ideals of R and $\mathcal{I}(R)$ respectively. Let $\mathfrak{M}(\mathcal{I}(R))$ denote the space of minimal prime ideals in $\mathcal{I}(R)$ endowed with the hull kernel topology. For this topology the set $\{U((a]) \,|\, a \in R\}$ will form a base for open sets where $U((a]) = \{P \in \mathfrak{M}(\mathcal{I}(R)) \,|\, (a] \notin \mathcal{P}\}$. This space is a compact Hausdorff space by [1, Lemma 1.4]. A sufficient condition for the space $\mathfrak{M}(R)$ to be a continuous image of the space $\mathfrak{M}(\mathcal{I}(R))$ is given in the following lemma.

Lemma 4.7. *Let R be an ADL such that $C(\mathcal{P}) \in \mathfrak{M}(R)$ for each $\mathcal{P} \in \mathfrak{M}(\mathcal{I}(R))$. Then the mapping $\phi : \mathfrak{M}(\mathcal{I}(R)) \to \mathfrak{M}(R)$ defined by $\phi(\mathcal{P}) = C(\mathcal{P})$, for each $\mathcal{P} \in \mathfrak{M}(\mathcal{I}(R))$ is an onto continuous closed mapping.*

Proof. Obviously, by the given condition ϕ is well defined map. Let $Q \in \mathfrak{M}(R)$. Then $\Gamma(Q)$ is a prime ideal in $\mathcal{I}(R)$ and hence it contains a minimal prime ideal $\mathcal{P}$ of $\mathcal{I}(R)$ (by Lemma 2.5). we claim that $C(\mathcal{P}) = Q$. Let $x \in C(\mathcal{P})$. Therefore $x \in J$ for some $J \in \mathcal{P}$. As $(x] \subseteq J$ and $J \in \mathcal{P}$ we get $(x] \in \mathcal{P}$, $\mathcal{P}$ being an ideal in $\mathcal{I}(R)$. Thus $(x] \in \Gamma(Q)$, since $\mathcal{P} \subseteq \Gamma(Q)$. But then $x \in Q$ and this shows that $C(\mathcal{P}) \subseteq Q$. Both $C(\mathcal{P})$ and Q being minimal prime ideals in R, we get $C(\mathcal{P}) = Q$. This in turn shows that ϕ is an onto mapping. Select any $a \in R$. Then

$$\begin{aligned}
\phi^{-1}[V(a)] &= \phi^{-1}\{M \in \mathfrak{M}(R) \,|\, a \notin M\} \\
&= \{\mathcal{P} \in \mathfrak{M}(\mathcal{I}(R)) \,|\, a \notin \phi(\mathcal{P})\} \\
&= \{\mathcal{P} \in \mathfrak{M}(\mathcal{I}(R)) \,|\, a \notin C(\mathcal{P})\} \\
&= \{\mathcal{P} \in \mathfrak{M}(\mathcal{I}(R)) \,|\, (a] \notin \mathcal{P}\} \\
&= U((a]).
\end{aligned}$$

Thus the inverse image of a basic open set in $\mathfrak{M}(R)$ is again a open set in $\mathfrak{M}(\mathcal{I}(R))$. Hence ϕ is a continuous map. The space $\mathfrak{M}(\mathcal{I}(R))$ is a compact space and $\mathfrak{M}(R)$ is a Hausdorff space. Hence the mapping ϕ being continuous, is a closed mapping. □

Using the continuity of the mapping ϕ defined in Lemma 4.7, we characterize a $*$-ADL as follows.

Theorem 4.8. *R is a $*$-ADL if and only if $C(\mathcal{P}) \in \mathfrak{M}(R)$ for each $\mathcal{P} \in \mathfrak{M}(\mathcal{I}(R))$.*

Proof. If Part : Let R be a $*$-ADL and let $\mathcal{P} \in \mathfrak{M}(\mathcal{I}(R))$. Select $x \in C(\mathcal{P})$. As $x \in J$ for some $J \in \mathcal{P}$ we get $(x] \subseteq J$. $\mathcal{P}$ being an ideal in $\mathcal{I}(R)$, we get $(x] \in \mathcal{P}$. Again, R being a $*$-ADL, there exists $x' \in R$ such that $x \wedge x' = 0$ and $x \vee x'$ is a dense element in R. But then $(x]\underline{\vee}(x'] = (x \vee x']$ is a dense element in $\mathcal{I}(R)$ and hence can not be contained in $\mathcal{P}$, $\mathcal{P}$ being a minimal prime ideal in $\mathcal{I}(R)$. Hence $(x'] \notin \mathcal{P}$ implies $x' \notin C(\mathcal{P})$. Thus given $x \in C(\mathcal{P})$, indeed there exists $x' \notin C(\mathcal{P})$ such that $x \wedge x' = 0$. Hence the prime ideal $C(\mathcal{P})$ is a minimal prime ideal of R (by Lemma 2.4).
Only if part : Assume that $C(\mathcal{P}) \in \mathfrak{M}(R)$ for each $\mathcal{P} \in \mathfrak{M}(\mathcal{I}(R))$. Then by Lemma 4.7, the mapping $\phi : \mathfrak{M}(\mathcal{I}(R)) \to \mathfrak{M}(R)$ defined by $\phi(\mathcal{P}) = C(\mathcal{P})$ is an onto continuous mapping. As $\mathfrak{M}(\mathcal{I}(R))$ is a compact space (by [1, Lemma 1.4]) it follows that $\mathfrak{M}(R)$ is a compact space. Hence R is a $*$-ADL by Lemma 2.13. □

If $\mathfrak{M}(R)$ and $\mathfrak{M}(\mathcal{I}(R))$ are homeomorphic, then R is a $*$-ADL (by proof of "only if part" of Theorem 4.8). Again by Theorem 4.8 (if part) and Lemma 4.7, we get for a $*$-ADL R the two spaces $\mathfrak{M}(R)$ and $\mathfrak{M}(\mathcal{I}(R))$ will be homeomorphic if the mapping ϕ defined in Lemma 4.7, is one-one i.e. for a given $Q \in \mathfrak{M}(R)$ there exists unique $\mathcal{P} \in \mathfrak{M}(\mathcal{I}(R))$ such that $C(\mathcal{P}) = Q$. We characterize this property as follows.

Theorem 4.9. *In an ADL R $J^{**} \in \mathcal{A}_0(R)$, for each $J \in \mathcal{I}(R)$, if and only if R is a $*$-ADL and for each $Q \in \mathfrak{M}(R)$ there exists unique $\mathcal{P}$ in $\mathfrak{M}(\mathcal{I}(R))$ such that $C(\mathcal{P}) = Q$.*

Proof. Let for each $J \in \mathcal{I}(R)$, $J^{**} \in \mathcal{A}_0(R)$. Select $x \in R$. Particularly taking $J = (x]^{**}$ we get $J^{**} = (x]^{**} \in \mathcal{A}_0(R)$. Hence there exists $x' \in R$ such that $(x]^{**} = (x']^{*}$. This shows that R is a $*$-ADL. Let $Q \in \mathfrak{M}(R)$ and $\mathcal{P}_1, \mathcal{P}_2 \in \mathfrak{M}(\mathcal{I}(R))$ such that $Q = C(\mathcal{P}_1) = C(\mathcal{P}_2)$. Let $J \in \mathcal{P}_1$. Then $J^{*} \in \mathcal{I}(R)$ and $J \cap J^{*} = (0]$. $\mathcal{P}_1 \in \mathfrak{M}(\mathcal{I}(R))$ and $J \in \mathcal{P}_1$ imply $J^{*} \notin \mathcal{P}_1$ (by Lemma 2.6). By the given condition , $[J^{*}]^{**}(= J^{*}) \in \mathcal{A}_0(R)$. Hence there exists $y \in R$ such that $J^{*} = (y]^{*}$. Again $\mathcal{P}_1 \in \mathfrak{M}(\mathcal{I}(R))$ and $J \in \mathcal{P}_1 \Rightarrow J^{**} \in \mathcal{P}_1$ (by Lemma 2.6). Hence $(y]^{**} \in \mathcal{P}_1$. As $(y] \subseteq (y]^{**} \in \mathcal{P}_1$ we get $(y] \in \mathcal{P}_1$. But then $y \in C(\mathcal{P}_1)$

and $y \in C(\mathcal{P}_2)$ as $C(\mathcal{P}_1) = C(\mathcal{P}_2)$. Therefore $y \in K$ for some $K \in \mathcal{P}_2$. Hence $(y] \subseteq K$, $K \in \mathcal{P}_2$ imply $(y] \in \mathcal{P}_2$, $\mathcal{P}_2$ being an ideal in $\mathcal{I}(R)$. But then $(y]^{**} \in \mathcal{P}_2$ as $\mathcal{P}_2 \in \mathfrak{M}(\mathcal{I}(R))$ (by Lemma 2.6). As $J \subseteq J^{**} = (y]^{**}$ we get $J \in \mathcal{P}_2$ as $(y]^{**} \in \mathcal{P}_2$ and $\mathcal{P}_2$ is an ideal in R. This shows that $\mathcal{P}_1 \subseteq \mathcal{P}_2$. Both being minimal prime ideals in $\mathcal{I}(R)$, we get $\mathcal{P}_1 = \mathcal{P}_2$. Thus for given $Q \in \mathfrak{M}(R)$, there exists unique $\mathcal{P} \in \mathfrak{M}(\mathcal{I}(R))$ such that $C(\mathcal{P}) = Q$.
Conversely, let R be an $*$-ADL such that for each $Q \in \mathfrak{M}(\mathcal{I}(R))$, there exists unique $\mathcal{P}$ in $\mathfrak{M}(\mathcal{I}(R))$ such that $C(\mathcal{P}) = Q$. This in turn gives that $C(\mathcal{P}_1) = C(\mathcal{P}_2) \Rightarrow \mathcal{P}_1 = \mathcal{P}_2$ for $\mathcal{P}_1, \mathcal{P}_2 \in \mathfrak{M}(\mathcal{I}(R))$, by Theorem 4.8. But then the mapping $\phi : \mathfrak{M}(\mathcal{I}(R)) \to \mathfrak{M}(R)$ defined by $\phi(\mathcal{P}) = C(\mathcal{P})$ is a homeomorphism (by Lemma 4.7 and Theorem 4.8). $\mathfrak{M}(\mathcal{I}(R))$ is an extremally disconnected space (by [1, Lemma 1.4]). But then $\mathfrak{M}(R)$ is also extremally disconnected. R being an $*$-ADL, $\mathfrak{M}(R)$ is a compact space (by Theorem **??**). Let $J \in \mathcal{I}(R)$. Then $h(J^*)$ is both open and closed in $\mathfrak{M}(R)$. Hence $\mathfrak{M}(R) \setminus h(J^*)$ being open in $\mathfrak{M}(R)$ can be expressed as a union of basic open sets. Again $\mathfrak{M}(R)$ is compact and $\mathfrak{M}(R) \setminus h(J^*)$ is a closed subset of $\mathfrak{M}(R)$ will imply $\mathfrak{M}(R) \setminus h(J^*)$ is itself compact. Hence $\mathfrak{M}(R) \setminus h(J^*) = \bigcup_{i=1}^{n} V(a_i)$. Define $y = \bigvee_{i=1}^{n} a_i$. Then $\mathfrak{M}(R) \setminus h(J^*) = V(y) = \mathfrak{M}(R) \setminus V((y]^*) = h((y]^*) = \mathfrak{M}(R) \setminus h(y) = \mathfrak{M}(R) \setminus h(\{y\}^{**})$. Hence $h(J^*) = h(\{y\}^{**})$ i.e. $k(h(J^*)) = k(h(\{y\}^{**}))$. Therefore $J^* = \{y\}^{**}$. Hence $J^{**} \in \mathcal{A}_0(R)$ □

The conjunction of Lemma 4.7, Theorem 4.8 and Theorem 4.9 yields a result analogous to [1, Theorem 2.3] proved by Cornish for a 0-distributive lattice.

Theorem 4.10. *Following statements are equivalent in R.*

1. *$\mathfrak{M}(R)$ is a compact, Hausdorff and extremally disconnected space.*
2. *The space $\mathfrak{M}(R)$ and $\mathfrak{M}(\mathcal{I}(R))$ are homeomorphic.*
3. *For each $J \in \mathcal{I}(R)$, $J^{**} \in \mathcal{A}_0(R)$.*

If the family $\{h(x) \mid x \in R\}$ is considered as an open basis for $\mathfrak{M}(R)$, the resulting topology is called the dual hull kernel topology. We denote by τ_h the hull kernel topology on $\mathfrak{M}(R)$ and by τ_d the dual hull kernel topology on $\mathfrak{M}(R)$. We know that $\{h(x) \mid x \in R\}$ is an open basis for the dual hull kernel topology. Now as $h(x) = \mathfrak{M}(R) \setminus V(x)$ for any x in R and by Theorem 3.2, $V(x)$ is a closed in $\mathfrak{M}(R)$ we get every basic open set in the dual hull kernel topology is open in the hull kernel topology. Hence τ_h is finer than τ_d.

As a consequence of the results proved in this section and in [11] we obtain

Theorem 4.11. *The following statements are equivalent in* R.

1. R *is a* $*$*-ADL.*
2. *The space* $\mathfrak{M}(R)$ *is compact.*
3. $\tau_h = \tau_d$.
4. $\{V(x) \mid x \in R\} = \{h(y) \mid y \in R\}$.
5. $< \{V(x) \mid x \in R\}, \cup, \cap >$ *is a Boolean lattice.*
6. R/θ *is a Boolean lattice, where* θ *is the congruence relation on* R *defined on* R *by* $x \equiv y(\theta)$ *if and only if* $\{x\}^* = \{y\}^* \, (x, y \in R)$.
7. $C(\mathcal{P}) \in \mathfrak{M}(R)$ *for each* $\mathcal{P} \in \mathfrak{M}(\mathcal{I}(R))$.

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