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Characterizing Completely Multiplicative Polynomial-Arithmetic Functions by Generalized Möbius Functions

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Abstract: Let Ω denote the set of monic polynomials over a finite field and let $\mathcal{A}(\Omega)$ be the ring of arithmetic functions $f : \Omega \to \mathbb{C}$. We construct a generalized Möbius functions in $\mathcal{A}(\Omega)$ and use it to characterize completely multiplicative functions in $\mathcal{A}(\Omega)$.

Keywords: polynomial-arithmetic function, complete multiplicativity, generalized Möbius function

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1 Introduction

In the classical case, Hsu [2], see also [9], [1], introduced the *SHM (Souriau-Hsu-Möbius) function*

$$\mu_\alpha(n) = \prod_{p|n} \binom{\alpha}{\nu_p(n)} (-1)^{\nu_p(n)},$$

where $\alpha \in \mathbb{R}$, and $n = \prod_{p \text{ prime}} p^{\nu_p(n)}$ denotes the unique prime factorization of $n \in \mathbb{N}$, $\nu_p(n)$ being the largest exponent of the prime p that divides n. This function generalizes the usual Möbius function, μ, because $\mu_1 = \mu$. Using the

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generalized Möbius function, Laohakosol et al [4], gave two characterizations of completely multiplicative functions. Save a minor condition, they read $(\mu_\alpha f)^{-1} = \mu_{-\alpha} f$ and $f^\alpha = \mu_{-\alpha} f$, where f^α is the α^{th} power function.

By a polynomial-arithmetic function, [8], we mean a mapping f from the set, Ω, of all monic polynomials over a finite field $\mathbb{F}_{p^n}$, where p is a prime and $n \in \mathbb{N}$ [7], into the field of complex numbers $\mathbb{C}$. Let $(\mathcal{A}(\Omega), +, *)$ denote the set of all polynomial-arithmetic functions equipped with addition and Dirichlet convolution defined over Ω, respectively, by

$$(f+g)(M) = f(M) + g(M)$$
$$(f*g)(M) = \sum_{D|M}^{(\Omega)} f(D)\, g\left(\frac{M}{D}\right)$$

for all $M \in \Omega$, where the summation is over all $D \in \Omega$ which are divisors of M. As in the case of classical arithmetic functions, it is easy to check that $(\mathcal{A}(\Omega), +, *)$ is an integral domain with identity I_Ω ([8]), defined by

$$I_\Omega(M) = \begin{cases} 1 & \text{if} \quad M = 1_\Omega \\ 0 & \text{otherwise,} \end{cases}$$

where 1_Ω is the identity element in $\mathbb{F}_{p^n}$.

Throughout, the notation $\sum^{(\Omega)}$ signifies a summation taken over monic polynomials in Ω.

The polynomial-Möbius function is defined, [3], by

$$\mu^\Omega(M) = \begin{cases} 1 & \text{if } M = 1_\Omega, \\ 0 & \text{if } P^2 | M,\ P \text{ irreducible element of } \Omega, \\ (-1)^t & \text{if } M = P_1 P_2 \cdots P_t,\ \text{a product of distinct irreducible} \\ & \text{elements of } \Omega. \end{cases}$$

A function $f \in \mathcal{A}(\Omega)$ is said to be *multiplicative* if

$$f(MN) = f(M)\, f(N) \tag{1}$$

whenever $(M, N) = 1_\Omega$ and f is said to be *completely multiplicative* if (1) holds for all pairs of polynomials M, N [8]. Further, $f(1_\Omega) = 1$ if f is multiplicative. It is clear that μ^Ω is multiplicative.

The objective of this paper is to construct generalized polynomial-Möbius functions and establish some characterizations of completely multiplicative functions in $\mathcal{A}(\Omega)$ using these functions.

2 Preliminaries

We have shown in [3], that the set

$$\mathcal{U}(\Omega) := \{f \in \mathcal{A}(\Omega) : f(1_\Omega) \neq 0\}$$

is the set of all units in $\mathcal{A}(\Omega)$. That is, for every $f \in \mathcal{U}(\Omega)$, there is $f^{-1} \in \mathcal{A}(\Omega)$, the inverse of f with respect to the Dirichlet convolution, such that $f * f^{-1} = I_\Omega$ and

$$f^{-1}(1_\Omega) = \frac{1}{f(1_\Omega)}, f^{-1}(M) = \frac{-1}{f(1_\Omega)} \sum_{D|M, D \neq 1_\Omega}^{(\Omega)} f(D)\, g\left(\frac{M}{D}\right) \qquad (M \in \Omega \setminus \{1_\Omega\}).$$

It is easy to see that $(\mathcal{U}(\Omega), *)$ is an abelian group with identity I_Ω and the set of multiplicative functions forms a subgroup of $\mathcal{U}(\Omega)$. Note that $u^{-1} = \mu^\Omega$, [8], where u is a unit function $(u(M) = 1 \;\; M \in \Omega)$.

An arithmetic function $a \in \mathcal{A}(\Omega)$ is *completely additive* if $a(MN) = a(M) + a(N)$ for all $M, N \in \Omega$ [3]. Note that if $a \in \mathcal{A}(\Omega)$ is completely additive, then $a(1_\Omega) = 0$.

Let

$$\mathcal{A}_1(\Omega) = \{f \in \mathcal{A}(\Omega) : f(1_\Omega) \in \mathbb{R}\} \text{ and } \mathcal{P}(\Omega) = \{f \in \mathcal{A}(\Omega) : f(1_\Omega) > 0\} \subseteq \mathcal{U}(\Omega).$$

Definition 2.1. ([3]) Let $a \in \mathcal{A}(\Omega)$ be a completely additive arithmetic function for which $a(M) \neq 0$ for all $M \in \Omega \setminus \{1_\Omega\}$. The polynomial-logarithmic operator (associated with a) is the map $Log_\Omega : \mathcal{P}(\Omega) \to \mathcal{A}_1(\Omega)$, defined by

$$\begin{aligned} Log_\Omega\, f(1_\Omega) &= log f(1_\Omega), \\ Log_\Omega\, f(M) &= \frac{1}{a(M)} \sum_{D|M}^{(\Omega)} f(D)\, f^{-1}\left(\frac{M}{D}\right) a(D) \end{aligned} \tag{2}$$

for all $M \in \Omega \setminus \{1_\Omega\}$ where the right-hand side of the first equation denotes the real logarithmic value.

In the classical case, this logarithmic operator was first introduced by Rearick ([5],[6]). We have shown in [3], that Log_Ω is a bijection of $\mathcal{P}(\Omega)$ onto $\mathcal{A}_1(\Omega)$ and

$$Log_\Omega(f * g) = Log_\Omega\, f + Log_\Omega\, g \qquad (f, g \in \mathcal{A}(\Omega)). \tag{3}$$

Therefore, it is possible to define a polynomial-exponential operator

$$Exp_\Omega : \mathcal{A}_1(\Omega) \to \mathcal{P}(\Omega)$$

as $Exp_\Omega = (Log_\Omega)^{-1}$. For $f \in \mathcal{P}(\Omega)$ and $\alpha \in \mathbb{R}$, the α^{th} *polynomial-power function* is defined as

$$f^\alpha = Exp_\Omega(\alpha Log_\Omega\, f). \tag{4}$$

Clearly, $f^0 = I_\Omega$ and $f^1 = f$. For $r \in \mathbb{N}$, using (3) and (4), we obtain

$$\begin{aligned} f^r &= Exp_\Omega(rLog_\Omega\, f) \\ &= Exp_\Omega(Log_\Omega\, f + \cdots + Log_\Omega f) \\ &= Exp_\Omega(Log_\Omega\,(f * \cdots * f)) \\ &= f * \cdots * f \quad (r \text{ factors}). \end{aligned} \tag{5}$$

We can show similarly that

$$f^{-r} = f^{-1} * f^{-1} * \cdots * f^{-1} \qquad (r \text{ factors}).$$

Let a be a completely additive arithmetic function for which $a(M) \neq 0$ for all $M \in \Omega \backslash \{1_\Omega\}$. It follows from (2) that

$$aLog_\Omega\, f = f^{-1} * fa. \tag{6}$$

If we replace f with $Exp_\Omega\, f$ in (6), we obtain

$$aExp_\Omega\, f = Exp_\Omega f * fa.$$

Therefore $Exp_\Omega f$ is uniquely determined by the formulas

$$\begin{aligned} &Exp_\Omega f(1_\Omega) = exp(f(1_\Omega)), \\ &Exp_\Omega f(M) = \frac{1}{a(M)} \sum_{D|M}^{(\Omega)} Exp_\Omega\, f(D)\, f\left(\frac{M}{D}\right) a\left(\frac{M}{D}\right) \end{aligned} \tag{7}$$

for all $M \in \Omega \backslash \{1_\Omega\}$. From (4), (7) and (2), it is not difficult to prove that for fixed $M \in \Omega$, the expression $f^\alpha(M) = Exp_\Omega(\alpha Log_\Omega\, f)(M)$ is a polynomial in α.

3 Main Results

It is well-known that, each nonconstant monic polynomial $M \in \Omega$ can be uniquely written in the form

$$M = P_1^{a_1} P_2^{a_2} \cdots P_k^{a_k},$$

where $P_1, P_2, \ldots, P_k$ are monic irreducible polynomials over $\mathbb{F}_{p^n}$ and $a_1, a_2, \ldots, a_k$, $k \in \mathbb{N}$ [7]. For $\alpha \in \mathbb{R}$, define $\mu_\alpha^\Omega : \Omega \to \mathbb{C}$ by

$$\mu_\alpha^\Omega(M) = \prod_{i=1}^{k} \binom{\alpha}{a_i}(-1)^{a_i}, \quad \mu_\alpha^\Omega(1_\Omega) = 1, \tag{8}$$

where

$$\binom{\alpha}{0} = 1, \ \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \qquad (n \in \mathbb{N}). \tag{9}$$

This function is called the *polynomial SHM function* because $\mu_1^\Omega = \mu^\Omega$, the polynomial-Möbius function. Observe that $\mu_0^\Omega = I$ and $\mu_{-1}^\Omega = u$. It is clear by the definition of μ_α^Ω that μ_α^Ω is multiplicative for all real number α. It follows that $\mu_\alpha^\Omega * \mu_\beta^\Omega = \mu_{\alpha+\beta}^\Omega$ for all real numbers α and β.

We first recall two propositions in [3]:

Proposition 3.1. *[3] Let $f \in \mathcal{A}(\Omega)$ be multiplicative. Then f is completely multiplicative if and only if $f^{-1}(M) = f\mu^\Omega(M)$ for all $M \in \Omega$.*

Proposition 3.2. *[3] If $f \in \mathcal{A}(\Omega)$ is multiplicative, then f is completely multiplicative if and only if $f\left(P^k\right) = f(P)^k$ for all irreducible polynomials $P \in \Omega$ and for all $k \in \mathbb{N}$.*

Now for the main results, we prove the following lemma.

Lemma 3.3. *A multiplicative function f is completely multiplicative if and only if $f(g*h) = fg * fh$ for all $g,\ h \in \mathcal{A}(\Omega)$.*

Proof. If f is completely multiplicative, then for all $g, h \in \mathcal{A}(\Omega)$ and all $M \in \Omega$, we have

$$\begin{aligned} f(g*h)(M) &= f(M) \sum_{D|M}^{(\Omega)} g(D)h(M/D), \\ &= \sum_{D|M}^{(\Omega)} f(D)g(D)f(M/D)h(M/D), \\ &= \sum_{D|M}^{(\Omega)} fg(D)fh(M/D), \\ &= (fg * fh)(M). \end{aligned}$$

Conversely, assume that $f(g * h) = fg * fh$ for all $g,\ h \in \mathcal{A}(\Omega)$. Then the equation holds when $g = u$ and $h = \mu^{\Omega}$ i.e.

$$I_{\Omega} = fI_{\Omega} = f(u * \mu^{\Omega}) = fu * f\mu^{\Omega} = f * f\mu^{\Omega}.$$

This implies $f^{-1} = f\mu^{\Omega}$ and the desired result follows by Proposition 3.1. □

Our first main result reads:

Theorem 3.4. *Let $f \in \mathcal{A}(\Omega)$ be a multiplicative function and α a nonzero real number. Then f is completely multiplicative if and only if $(\mu^{\Omega}_{\alpha} f)^{-1} = \mu^{\Omega}_{-\alpha} f$.*

Proof. Let $f \in \mathcal{A}(\Omega)$ be a multiplicative function and $\alpha \in \mathbb{R}\backslash\{0\}$. If f is completely multiplicative, then

$$\mu^{\Omega}_{\alpha} f * \mu^{\Omega}_{-\alpha} f = (\mu^{\Omega}_{\alpha} * \mu^{\Omega}_{-\alpha}) f = \mu^{\Omega}_{0} f = I_{\Omega}\ f = I_{\Omega},$$

by Lemma 3.3 and so $(\mu^{\Omega}_{\alpha} f)^{-1} = \mu^{\Omega}_{-\alpha} f$.

Conversely, assume that $(\mu^{\Omega}_{\alpha} f)^{-1} = \mu^{\Omega}_{-\alpha} f$. By Proposition 3.2, it suffices to show that $f\left(P^{k}\right) = f\left(P\right)^{k}$ for all irreducible $P \in \Omega$ and for all $k \in \mathbb{N}$. Since this is trivial for $k = 1$, we consider $k \geq 2$. We proceed by induction assuming that $f\left(P^{i}\right) = f\left(P\right)^{i}$ holds for $i \in \{1, 2, \ldots, k-1\}$. Rewriting hypothesis in an equivalent form as

$$\mu^{\Omega}_{\alpha} f * \mu^{\Omega}_{-\alpha} f = I_{\Omega}$$

and evaluating at P^{k}, we get

$$\begin{aligned} 0 &= I_{\Omega}(P^{k}) = (\mu^{\Omega}_{\alpha} f * \mu^{\Omega}_{-\alpha} f)(P^{k}), \\ &= \sum_{i+j=k} \mu^{\Omega}_{-\alpha} f(P^{i}) \mu^{\Omega}_{\alpha} f(P^{j}), \\ &= \sum_{i+j=k} \binom{-\alpha}{i} (-1)^{i} f(P^{i}) \binom{\alpha}{j} (-1)^{j} f(P^{j}), \\ &= (-1)^{k} \sum_{i+j=k} \binom{-\alpha}{i} \binom{\alpha}{j} f(P^{i}) f(P^{j}). \end{aligned}$$

From $(1+z)^{\alpha}(1+z)^{-\alpha} = 1$, we infer that,

$$\sum_{i+j=k} \binom{-\alpha}{i} \binom{\alpha}{j} = 0,$$

which implies that

$$\binom{-\alpha}{k} + \binom{\alpha}{k} = -\left[\sum_{i=1}^{k-1} \binom{-\alpha}{i}\binom{\alpha}{k-i}\right]. \tag{10}$$

Using induction hypothesis, we get

$$0 = \left[\binom{-\alpha}{k} + \binom{\alpha}{k}\right] f(P^k) + \left[\sum_{i=1}^{k-1} \binom{-\alpha}{i}\binom{\alpha}{k-i}\right] f(P)^k$$

and so $f\left(P^k\right) = f\left(P\right)^k$ follows from (10), $\alpha \neq 0$ and $k \geq 2$. □

Our last main result reads:

Theorem 3.5. *Let f be a multiplicative function and $\alpha \in \mathbb{R}$. Then*

(i) If f is completely multiplicative then $f^\alpha = \mu^\Omega_{-\alpha} f$.

(ii) For $\alpha \notin \{0,1\}$, if $f^\alpha = \mu^\Omega_{-\alpha} f$, then f is completely multiplicative

Proof. (i) If f is completely multiplicative, then by Lemma 3.3 and (5), we have

$$f^r = f{*}f{*}\cdots{*}f = (u{*}u{*}\cdots{*}u)f = (\mu^\Omega_{-1}{*}\mu^\Omega_{-1}{*}\cdots{*}\mu^\Omega_{-1})f = \mu^\Omega_{-r} f \quad (r \in \mathbb{N}). \tag{11}$$

Let $M \in \Omega$ be fixed. From (4), (7) and (2), we can prove that $f^\alpha(M)$ is a polynomial in α and by (8) and (9), $(\mu^\Omega_{-\alpha} f)(M)$ is also a polynomial in α. Using (11), we have that $f^\alpha(M) = (\mu^\Omega_{-\alpha} f)(M)$ holds for infinitely many values of α. It follows that $f^\alpha(M) - (\mu^\Omega_{-\alpha} f)(M)$ is the zero polynomial and so $f^\alpha = \mu^\Omega_{-\alpha} f$ for all real α.

(ii) Let $\alpha \in \mathbb{R}\backslash\{0,1\}$. Since f is multiplicative, by Proposition 3.2, it suffices to show that $f\left(P^k\right) = f\left(P\right)^k$ for all irreducible polynomials $P \in \Omega$ and for all $k \in \mathbb{N}$. The case $k = 1$ being trivial. We proceed by induction assuming that $f\left(P^i\right) = f\left(P\right)^i$ holds for $i \in \{1, 2, \ldots, k-1\}$ $(k \geq 2)$.

We pause to prove an auxilliary claim.

Claim.

$$f^\alpha(P^k) = f(P)^k \left(\mu^\Omega_{-\alpha}(P^k) - \alpha\right) + \alpha f(P^k). \tag{12}$$

Proof of Claim. Let $r \in \mathbb{N}$. Then, using the induction hypothesis, we get

$$\begin{aligned} f^r(P^k) &= \sum_{i_1+\cdots+i_r=k} f(P^{i_1})\cdots f(P^{i_r}), \\ &= \sum_{i_1+\cdots+i_r=k,\ \text{all}\ i_j\neq k} f(P^{i_1})\cdots f(P^{i_r}) + rf(P^k), \\ &= f(P)^k \sum_{i_1+\cdots+i_r=k,\ \text{all}\ i_j\neq k} 1 + rf(P^k), \\ &= f(P)^k \left[\binom{r+k-1}{k} - r\right] + rf(P^k), \\ &= f(P)^k \left[(-1)^k\binom{-r}{k} - r\right] + rf(P^k), \\ &= f(P)^k \left(\mu^{\Omega}_{-r}(P^k) - r\right) + rf(P^k). \end{aligned} \tag{13}$$

From the useful fact, mentioned in the preliminaries, we known that the expression $f^{\alpha}(P^k)$ is a polynomial in α. By (8) and (9), the right hand side of (12) is a polynomial in α. Using (13), we obtain $f^{\alpha}(P^k) = f(P)^k\left(\mu^{\Omega}_{-\alpha}(P^k) - \alpha\right) + \alpha f(P^k)$ holds for all positive integer α. It follows that $f^{\alpha}(P^k) = f(P)^k\left(\mu^{\Omega}_{-\alpha}(P^k) - \alpha\right) + \alpha f(P^k)$ holds for all real numbers α.

Returning to the hypothesis, using (12) and evaluating at P^k, we get

$$\mu^{\Omega}_{-\alpha}(P^k) f(P^k) = f^{\alpha}(P^k) = f(P)^k\left(\mu^{\Omega}_{-\alpha}(P^k) - \alpha\right) + \alpha f(P^k)$$

and so

$$\left(\mu^{\Omega}_{-\alpha}(P^k) - \alpha\right) f(P^k) = \left(\mu^{\Omega}_{-\alpha}(P^k) - \alpha\right) f(P)^k.$$

Simplifying, we arrive at

$$\left[\binom{\alpha+k-1}{k} - \alpha\right] f(P^k) = \left[\binom{\alpha+k-1}{k} - \alpha\right] f(P)^k.$$

Since $\alpha \notin \{0,1\}$ and $k \geq 2$, then $\binom{\alpha+k-1}{k} - \alpha \neq 0$, and we have the result.

□

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