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# On a Generalization of Quasiposinormal Operators

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**Abstract:** Paper describes some properties for the operators $A$ on a Hilbert space $\mathcal{H}$ satisfying $(A^*A)^k \leq c^2 A^{*k}A^k$ for some $c > 0$, $k \geq 2$ and also presents some characterizations for the composition operators and the weighted composition operators on the Hilbert space $L^2$ to be of this type.

**Keywords:** hyponormal operator, posinormal operator, k-quasiposinormal operator and weighted composition operator

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## 1 Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space. The algebra of all operators on $\mathcal{H}$ is denoted by $\mathfrak{B}(\mathcal{H})$ and the symbols $Ran(A)$ and $Ker(A)$ are used to denote the range and kernel of an operator $A$ acting on $\mathcal{H}$ respectively. Throughout the paper, by an operator we mean a bounded linear transformation acting on a Hilbert space. Recall that an operator $A \in \mathfrak{B}(\mathcal{H})$, where $A^*$ stands for the adjoint of $A$, is said to be
*hyponormal* if $AA^* \leq A^*A$;
*quasihyponormal* if $A^*(AA^*)A \leq A^*(A^*A)A$ equivalently $(A^*A)^2 \leq A^{*2}A^2$;
*posinormal* if $AA^* \leq c^2A^*A$ for some $c > 0$;
*quasiposinormal* if $A^*(AA^*)A \leq c^2A^*(A^*A)A$ equivalently $(A^*A)^2 \leq c^2A^{*2}A^2$

for some $c > 0$.

The hyponormal, posinormal, quasihyponormal, and quasiposinormal classes of operators are discussed by many authors and we refer to [1,2,5,7,12,13] for more details and the applications of these classes of operators. The following relations with strict inclusion are well known.

$$\textit{hyponormal} \subset \textit{quasihyponormal}.$$

$$\textit{hyponormal} \subset \textit{posinormal} \subset \textit{quasiposinormal}.$$

The quasihyponormal class is generalized to (p,k)-quasihyponormal class [6], namely, $A$ satisfying $A^{*k}(AA^*)^pA^k \leq A^{*k}(A^*A)^pA^k$ and in [9] the quasiposinormal class is generalized to (p,k)-quasiposinormal class of operators, namely, $A$ satisfying $A^{*k}(AA^*)^pA^k \leq c^2A^{*k}(A^*A)^pA^k$, where $k$ is a positive integer and $0 < p \leq 1$. In [10], Patel has discussed some properties for a class of operators $A$ on a Hilbert space $\mathcal{H}$ satisfying $(A^*A)^k \leq A^{*k}A^k$, $k \geq 2$, which is named as (M,k) class. It is evident that for $k = 2$, the operators of class (M,k) become the class of quasihyponormal operators. The motive of this paper is twofold. First we introduce Posi-(M,k) operators and present some properties along with certain equivalent conditions for an operator to be Posi-(M,k). Strict inclusion of (M,k) class of operators in Posi-(M,k) class is also shown. Next we focus (in sections 2 and 3) on deriving conditions for composition and weighted composition operators on $L^2(\Omega, \mathcal{A}, \mu)$ to be in Posi-(M,k) class.

## 2 Generalizations

We begin with the following definition:

**Definition 2.1.** An operator $A \in \mathfrak{B}(\mathcal{H})$ is said to be Posi-(M,k) if $(A^*A)^k \leq c^2A^{*k}A^k$, $(k \geq 2)$, for some $c > 0$.

The collection of all Posi-(M,k) operators is referred as Posi-(M,k) class. It is interesting to note, similar to the fact that the (M,2) class of operators coincides to the class of quasihyponormal operators, the Posi-(M,2) class of operators coincides to the class of quasiposinormal operators. Consider the Hilbert space $\ell^2$ with standard orthonormal basis $\{e_n | n \geq 0\}$. We recall that a unilateral weighted shift $A$ on $\ell^2$ with weight $\langle\alpha_n\rangle_{n\geq 0}$ is injective if and only if the weight sequence $\langle\alpha_n\rangle_{n\geq 0}$ has no zero term. Let $A$ be the unilateral weighted shift with weighted sequence $\langle\alpha_n\rangle_{n\geq 0}$, where

$$\alpha_0=\alpha_1 = 0\ ,\quad \alpha_2=2 \text{ and } \ \alpha_n=1 \text{ if } n\geq 3.$$

Then $A$ is of Posi-(M,2) class with $(A^*A)^2 \leq 4A^{*2}A^2$. Also,

$$\langle (A^*A)^2e_2, e_2\rangle = 16 \text{ and } \langle (A^{*2}A^2)e_2, e_2\rangle = 4.$$

Hence $A$ is not of (M,2) class. This justifies the strict inclusion of (M,2) class of operators in Posi-(M,2)class.

For any positive integer $k\geq 2$, every operator of (M,k) class is of Posi-(M,k) class but the converse is not true. For, if we consider the unilateral weighted shift $A_k$ with weighted sequence $\langle\alpha_n\rangle_{n\geq 0}$, where

$$\alpha_n = 0 \text{ if } n<k,$$
$$\alpha_n \leq \alpha_{n+1} \text{ if } n>k$$

and $\alpha_k$ is taken such that $\alpha_k\geq \alpha_{2k-1}$. Then $A_k$ is of Posi-(M,k) class but not of (M,k) class. Clearly $A$ is not injective.

However, we note the following property, which is easy to prove:

*An injective unilateral weighted shift with weight $\langle\alpha_n\rangle_{n\geq 0}$ belongs to Posi-(M,k) class if and only if*

$$\sup_n \frac{|\alpha_n|^{k-1}}{|\alpha_{n+1}\alpha_{n+2}\cdots\alpha_{n+k-1}|}<\infty. \tag{2.1.1}$$

It can be easily seen that the condition (2.1.1) holds if a sequence $\langle\alpha_n\rangle_{n\geq 0}$ of nonzero terms converges to a nonzero number but (2.1.1) may fail to hold even if $\langle\alpha_n\rangle$ tends to zero ( e.g., condition (2.1.1) does not hold for $\alpha_n=\frac{1}{n(n-1)(n-2)\cdot\cdot 1}$ but holds for $\alpha_n=\frac{1}{n}$).

The following conclusion can be made by using [7, Remark page 4]:

*For an injective unilateral weighted shift $A$ with weight $\langle\alpha_n\rangle_{n\geq 0}$, following are equivalent*

1. *$A$ belongs to Posi-(M,2) class.*
2. $\sup_n \frac{|\alpha_n|}{|\alpha_{n+1}|}<\infty$.
3. *$A$ is posinormal.*

If $A = U|A|$ is the polar decomposition of an operator $A$ on a Hilbert space $H$ then $A$ injective implies that $|A|$ is injective and hence $|A|^n$ is injective for each natural number $n$. As a consequence $(A^*A)^n$ is injective for each natural number $n$. Whereas injectiveness of $A$ is obvious from the injectiveness of $A^*A$. Thus we have the following:

*An operator $A$ on a Hilbert space $H$ is injective if and only if $(A^*A)^k$ is injective for each natural number $k$.*

We use this fact to obtain the following result.

**Theorem 2.2.** *If $A \in \mathfrak{B}(\mathcal{H})$ is of Posi-(M,k) class then $Ker(A^k) = Ker(A)$.*

An immediate consequence of this theorem (which is also proved by an alternate way in corollary 2.11) is the following:

**Corollary 2.3.** *If $A \in \mathfrak{B}(\mathcal{H})$ is of Posi-(M,k) class then $Ker(A^{(k+1)}) = Ker(A^2)$.*

The next theorem presents some characterizations for an operator $A$ acting on a Hilbert space $H$ to be of class Posi-(M,k) for $k \geq 2$.

**Theorem 2.4.** *For an operator $A \in \mathfrak{B}(\mathcal{H})$, the following are equivalent:*

1. *$A$ is of Posi-(M,k) class.*
2. *There exists a positive operator $P \in \mathfrak{B}(\mathcal{H})$ satisfying*
$$(A^*A)^k = A^{*k}PA^k.$$
3. *There exists a positive operator $P \in \mathfrak{B}(\mathcal{H})$ satisfying*
$$(A^*A)^k \leq A^{*k}PA^k.$$
4. *There exists $C \in \mathfrak{B}(\mathcal{H})$ satisfying $|A|^k = A^{*k}C$, where $|A| = \sqrt{A^*A}$.*
5. *$Ran(|A|^k) \subseteq Ran(A^{*k})$.*

*Proof.* The proof follows using the ideas from [4, Theorem 1] given by Douglas. □

**Corollary 2.5.** *If $A \in \mathfrak{B}(\mathcal{H})$ is invertible then $A$ is of Posi-(M,k) class for each positive integer $k \geq 2$.*

*Proof.* In this case $Ran(|A|^k) = Ran(A^{*k}) = \mathcal{H}$. □

**Corollary 2.6.** *If $A \in \mathfrak{B}(\mathcal{H})$ is of Posi-(M,k) class and $V \in \mathfrak{B}(\mathcal{H})$ is an isometry then $VAV^*$ is also of Posi-(M,k) class.*

*Proof.* If $P$ is a positive operator satisfying the condition (2) of the Theorem 2.5 for the operator $A$ then $VPV^*$ is a positive operator satisfying the same condition for the operator $VAV^*$. □

Posi-(M,k) operators are not closed under translations and the adjoint of a Posi-(M,k) operator may not be Posi-(M,k). It can be verified by the facts that $U$ and $A = (U^* - 2I)$ are of Posi-(M,k) class because $U$ satisfies the condition (5) of the Theorem 2.5 and $A = (U^* - 2I)$ is invertible, where $U$ is the unilateral shift operator on the Hilbert space $\ell^2$. But $A + 2I = U^*$ is not of Posi-(M,k) class as

$$\big\langle (UU^*)^k e_1, e_1 \big\rangle = 1 \text{ and } \big\langle (U^k U^{*k}) e_1, e_1 \big\rangle = 0$$

where $e_1 =< 0, 1, 0, 0, 0, \cdots >\in \ell^2$. Evidentally, the sum of two operators of Posi-(M,k) class need not belongs to the same class. However, it is easy to verify that if $A \in \mathfrak{B}(\mathcal{H})$ is of Posi-(M,k) class then $\alpha A$ is of Posi-(M,k) class, for each $\alpha \in \mathbb{C}$.

It is also seen that the product $AB$ of two operators $A$ and $B$ of Posi-(M,k) class need not be in the Posi-(M,k) class. For, consider the unilateral shift operator $A$ and a diagonal operator $B$ with diagonal entries

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n = 1, \\ 1, & n \geq 2. \end{cases}$$

Then A and B both are of Posi-(M,2) class. AB is unilateral shift with weights $\beta_0 = 1, \beta_1 = 0$ and $\beta_n = 1$ for $n \geq 2$. Now

$$\big\langle ((AB)^* AB)^2 e_0, e_0 \big\rangle = 1 \text{ and } \big\langle ((AB)^{*2} (AB)^2) e_0, e_0 \big\rangle = 0.$$

Hence AB does not belong to Posi-(M,2) class.

In the next result, we present a sufficient condition for the product $AB$ in Posi-(M,k) class.

**Theorem 2.7.** *If $A$ and $B$ are of Posi-(M,k) class such that $A$ commutes with $B$ and $B^*$ then $AB$ is of Posi-(M,k) class.*

*Proof.* Suppose that

$$(A^*A)^k \leq c_1^2 A^{*k} A^k$$

and

$$(B^*B)^k \leq c_2^2 B^{*k} B^k$$

for some $c_1, c_2 > 0$. The positive operators $\big(c_1^2 A^{*k} A^k - (A^*A)^k\big)$ and $\big(c_2^2 B^{*k} B^k - (B^*B)^k\big)$ commute, hence

$$\big(c_1^2 A^{*k} A^k - (A^*A)^k\big)\big(c_2^2 B^{*k} B^k + (B^*B)^k\big) \geq 0 \qquad (2.4.1).$$

By the similar argument, we have

$$\big(c_1^2 A^{*k} A^k + (A^*A)^k\big)\big(c_2^2 B^{*k} B^k - (B^*B)^k\big) \geq 0 \qquad (2.4.2).$$

Using (2.4.1) and (2.4.2), we find that

$$\begin{aligned} \big((AB)^*(AB)\big)^k &= \big(A^*A\big)^k \big(B^*B\big)^k \\ &\leq c^2 \big(A^{*k}A^k\big)\big(B^{*k}B^k\big) \\ &= c^2 \big(AB)^{*k}(AB)^k\big), \end{aligned}$$

where $c = c_1 c_2$. Hence $AB$ is of Posi-(M,k) class. □

It is not known whether the product $AB$ of two commuting operators $A$ and $B$ of Posi-(M,k) class belongs to Posi-(M,k) class. However, we have the following.

**Corollary 2.8.** *If $A$ is of Posi-(M,k) class and $B$ is a normal operator such that $A$ commutes with $B$ then $AB$ is of Posi-(M,k) class.*

*Proof.* Proof follows immediately by applying Putnam-Fuglede Theorem [11]. □

Our next result needs the Hölder-McCarthy Inequality, which states the following.

Let $A$ be a positive operator on $\mathcal{H}$. Then the following hold:

1. : $\langle A^p x,\ x\rangle \leq \langle Ax,\ x\rangle^p \|x\|^{2(1-p)}$ if $0 < p \leq 1$.
2. : $\langle A^p x,\ x\rangle \geq \langle Ax,\ x\rangle^p \|x\|^{2(1-p)}$ if $p > 1$.

**Theorem 2.9.** *If $A$ is of Posi-(M,k) class then there exists $c > 0$ such that*

$$c\|Ax\|^{2(k-1)}\|A^{k+1}x\| \geq \|A^2x\|^{2k}$$

*for all $x \in H$.*

*Proof.* Suppose that $(A^*A)^k \le cA^{*k}A^k$ for some $c > 0$. The required inequality is trivially true if $Ax = 0$, so we may assume that $Ax \ne 0$. Then

$$\begin{aligned}
\|A^{k+1}x\|^2 &= \big\langle (A^{*k}A^k)(Ax), Ax \big\rangle \\
&\ge c^{-1}\big\langle (A^*A)^k(Ax), Ax \big\rangle \\
&\ge c^{-1}\|Ax\|^{-2(k-1)}\big\langle (A^*A)(Ax), Ax \big\rangle^k \\
&= c^{-1}\|Ax\|^{-2(k-1)}\|A^2x\|^{2k}.
\end{aligned}$$

Hence $c\|Ax\|^{2(k-1)}\|A^{k+1}x\|^2 \ge \|A^2x\|^{2k}$ for all $x \in H$. □

The following result is immediate from Theorem 2.9.

**Corollary 2.10.** *If* $A \in \mathfrak{B}(\mathcal{H})$ *is of Posi-(M,k) class then* $Ker(A^{(k+1)}) = Ker(A^2)$.

# 3 Composition Operators

Let $(\Omega, \mathcal{A}, \mu)$ be a $\sigma-$ finite measure space. A measurable transformation $T : \Omega \to \Omega$ satisfying

$$\mu(T^{-1}(B)) = 0 \;\text{ whenever }\; \mu(B) = 0 \;\text{ for }\; B \in \mathcal{A}$$

is said to be a non-singular measurable transformation. If $T$ is non-singular, then the measure $\mu T^{-1}$ given by

$$(\mu T^{-1})(B) = \mu(T^{-1}(B)) \;\text{ for }\; B \in \mathcal{A},$$

is absolutely continuous with respect to the measure $\mu$ and we denote it by writing $\mu T^{-1} \ll \mu$. Hence by the Radon-Nikodym theorem, there exists a non-negative measurable function $h$ such that

$$(\mu T^{-1})(B) = \int_B h d\mu \;\;,$$

for every $B \in \mathcal{A}$. The function $h$ is called the Radon-Nikodym derivative of the measure $\mu T^{-1}$ with respect to the measure $\mu$. It is denoted by $h = d\mu T^{-1}/d\mu$.

For $k \ge 1$, define $T^k = \underbrace{T \circ T \circ \cdot \;\; (k\ times) \;\;\; \cdot \circ T}$. Then the Radon-Nikodyn derivative of $\mu T^{-k}$ with respect to $\mu$ is denoted by $h_k$. It is easy to check that $h_k = h \cdot h \circ T^{-1} \cdot h \circ T^{-2} \cdots \cdot h \circ T^{-(k-1)}$. We use the symbol $E$, which denotes the conditional expectation operator $E(.|T^{-1}(\mathcal{A})) = E(f)$. We refer [3,8] as well as the references included therein, to study the basic properties of expectation operator.

Let $L^2 = L^2(\Omega, \mathcal{A}, \mu)$ denote the space of all complex-valued measurable function for which $\int_\Omega |f|^2 d\mu < \infty$. A composition operator on $L^2$, induced by a non-singular measurable transformation $T$, is denoted by $C_T$ and is given by

$$C_T f = f \circ T \quad \text{for each } f \in L^2.$$

Then for $f \in L^2$ and for any positive integer k, $C_T^k f = f \circ T^k$ and $C_T^{*k} f = h_k \cdot E(f) \circ T^{-k}$, where $h_k = d\mu T^{-k}/d\mu$.

Theorem 2.5, when combined with these properties of the composition operator $C_T$, takes the following form.

**Theorem 3.1.** *Let $C_T \in \mathfrak{B}(L^2)$. Then the following are equivalent:*

1. *$C_T$ is of Posi-(M,k) class.*
2. *There exists a constant $c > 0$ such that*

$$\|h^{k/2} \cdot f\| \leq c\|\sqrt{h_k} \cdot f\|$$

   *for each $f \in L^2$.*

3. *$h^k \leq c^2 h_k$ , for some $c > 0$.*

**Corollary 3.2.** *For $C_T \in \mathfrak{B}(L^2)$, following are equivalent:*

1. *$C_T$ is quasiposinormal.*
2. *$\|h \cdot f\| \leq c\|\sqrt{h_2} \cdot f\|$, for each $f \in L^2$ and for some constant $c > 0$.*
3. *$h^2 \leq c^2 h_2$ for some $c > 0$.*
4. *$h \leq c^2 h_T$ for some $c > 0$, where $h_T = d\mu T^{-2}/d\mu T^{-1}$.*

*Proof.* Proof follows by setting $k = 2$ in Theorem 3.1. □

The next theorem gives a characterization for the adjoint of a composition operator to be of Posi-(M,k) class, which follows without any extra efforts.

**Theorem 3.3.** *Let $C_T \in \mathfrak{B}(L^2)$. A necessary and sufficient condition for $C_T^*$ to be of Posi-(M,k) class is that, for each $f \in L^2$*

$$\langle (h \circ T)^k \cdot E(f), f \rangle \leq c^2 \langle h_k \circ T^k \cdot E(f), f \rangle$$

*for some constant $c > 0$.*

**Corollary 3.4.** *Let $C_T \in \mathfrak{B}(L^2)$. If $T^{-1}(\mathcal{A}) = \mathcal{A}$ then $C_T^*$ is Posi-(M,k) if and only if for some $c > 0$, $(h \circ T)^k \leq c^2 h_k \circ T^k$.*

**Corollary 3.5.** *Let $C_T \in \mathfrak{B}(L^2)$. If $T^{-1}(\mathcal{A}) = \mathcal{A}$ then $C_T^*$ is quasiposinormal if and only if for some constant $c > 0$, $(h \circ T)^2 \leq c^2 h_2 \circ T^2$.*

**Example 3.6.** Consider the composition operator $C_T$ on $L^2(\Omega)$, where $\Omega = \mathbb{R}$, the set of all real numbers, $\mu =$Lebesgue measure, $\mathcal{A} =$ $\sigma-$algebra of all Lebesgue measurable subsets of real numbers and $T : \Omega \mapsto \Omega$ is given by

$$T(x) = x + a$$

for each $x \in \Omega$, $a > 0$ is a fixed real number. Then $h \equiv 1$ and also for each positive integer $k \geq 2$, $h_k \equiv 1$. Hence, $C_T$ and $C_T^*$ both are of Posi-(M,k) class for each $k \geq 2$.

**Example 3.7.** Let $\Omega = [0, 1]$, $\mu =$Lebesgue measure and $\mathcal{A}$ be the $\sigma-$algebra of all Lebesgue measurable subsets of the interval [0,1]. Let $T : \Omega \mapsto \Omega$ be given by

$$T(x) = \sqrt{x}$$

for each $x \in \Omega$. The Radon-Nikodym derivative $h_k$ of $\mu T^{-k}$ with respect to $\mu$ is given by

$$h_k(x) = 2^k x^{2^k - 1}$$

for each $x \in \Omega$. The composition operator $C_T$ on $L^2(\Omega)$ induced by $T$ is not of Posi-(M,k) class for any $k \geq 2$.

**Example 3.8.** Let $\Omega = \mathbb{R}$, the set of all real numbers, $\mu =$Lebesgue measure and $\mathcal{A}$ be the $\sigma-$algebra of all Lebesgue measurable subsets of real numbers. Let $T : \Omega \mapsto \Omega$ be given by

$$T(x) = 2x$$

for each $x \in \Omega$. Then $T$ induces the composition operator $C_T$ on $L^2(\Omega)$. In this case $h \equiv 1/2$. For each positive integer $k \geq 2$, $T^k : \Omega \mapsto \Omega$ is given by $T(x) = 2^k x$ for each $x \in \Omega$ satisfies $h_k \equiv 1/2^k$. Moreover $T^{-1}(\mathcal{A}) = \mathcal{A}$ so that $C_T$ and $C_T^*$ both are of Posi-(M,k) class for each $k \geq 2$.

# 4 Weighted Composition Operators

Let $W = W_{(u,T)}$ denote the weighted composition operator on $L^2$ given by $(f \mapsto u \cdot f \circ T)$, induced by a complex-valued mapping $u$ on $\Omega$ and a measurable transformation $T : \Omega \mapsto \Omega$. The adjoint $W^*$ of the weighted composition operator $W$ is given by

$$W^* f = h \cdot E(u \cdot f) \circ T^{-1}$$

for each $f \in L^2$. In case $u = 1$ *a.e.* then $W$ becomes the composition operator $C_T$.

The following results can be achieved without any extra efforts.

**Theorem 4.1.** *Let $W \in \mathfrak{B}(L^2)$. Then $W$ is of Posi-(M,k) class if and only if there exists a constant $c > 0$ such that*

$$(h \cdot E(u^2) \circ T^{-1})^k \leq c^2 h_k \cdot E(u_k^2) \circ T^{-k},$$

*where $u_k = u \cdot (u \circ T) \cdot (u \circ T^2) \cdots \cdots (u \circ T^{(k-1)})$ and $h_k = \frac{d\mu T^{-k}}{d\mu}$.*

**Corollary 4.2.** *Let $W \in \mathfrak{B}(L^2)$. If $T^{-1}(\mathcal{A}) = \mathcal{A}$ then $W$ is of Posi-(M,k) class if and only if*

$$(h \cdot u^2 \circ T^{-1})^k \leq c^2 h_k \cdot u_k^2 \circ T^{-k}$$

*for some $c > 0$.*

If we put $k = 2$, we have the following:

**Corollary 4.3.** *Let $W \in \mathfrak{B}(L^2)$. If $T^{-1}(\mathcal{A}) = \mathcal{A}$ then $W$ is quasiposinormal if and only if*

$$(h \cdot u^2 \circ T^{-1})^2 \leq c^2 h_2 \cdot u_2^2 \circ T^{-2}$$

*for some $c > 0$.*

**Theorem 4.4.** *Let $W \in \mathfrak{B}(L^2)$. Then $W^*$ is of Posi-(M,k) class if and only if there exists a constant $c > 0$ satisfying*

$$\langle u \cdot E(u^2)^{k-1} \cdot (h \circ T)^k \cdot E(uf), f \rangle \leq c^2 \langle u_k \cdot h_k \circ T^k \cdot E(u_k f), f \rangle$$

*for each $f \in L^2$.*

**Corollary 4.5.** *Let $W \in \mathfrak{B}(L^2)$. If $T^{-1}(\mathcal{A}) = \mathcal{A}$ then $W^*$ is of Posi-(M,k) class if and only if $u^{2k} \cdot (h \circ T)^k \leq c^2 u_k^2 \cdot h_k \circ T^k$, for some $c > 0$.*

**Corollary 4.6.** *$W^*$ is quasiposinormal if and only if for some $c > 0$,*

$$u^4 \cdot (h \circ T)^2 \leq c^2 u_2^2 \cdot h_2 \circ T^2.$$

**Example 4.7.** Let $\Omega = \mathbb{R}$, the set of all real numbers, $\mu =$Lebesgue measure, $\mathcal{A} =$ $\sigma-$algebra of all Lebesgue measurable subsets of real numbers. Consider the mappings $T : \Omega \mapsto \Omega$ given by

$$T(x) = x + a$$

and $u : \Omega \mapsto \Omega$ given by

$$u(x) = b$$

for each $x \in \Omega$, $a, b > 0$ are fixed real numbers. Then $u$ and $T$ induce the weighted composition operator $W$ on $L^2(\Omega)$. Also,

$$(h \cdot u^2 \circ T^{-1})^k = h_k \cdot u_k^2 \circ T^{-k} = b^{2k}$$

and

$$u^{2k} \cdot (h \circ T)^k = u_k^2 \cdot h_k \circ T^k = b^{2k}$$

so that $W$ and $W^*$ both are of Posi-(M,k) class for each $k \geq 2$.

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# References

[1] J.T. Campbell and W.E. Hornor, Seminormal composition operators, *J. Operator Theory*, **29**(1993), 323–343.

[2] J.T. Campbell and B.C. Gupta, On k-quasihyponormal operators, *Math. Japon.*, **23**(1978), 185–189.

[3] G. Datt, On k-quasiposiponormal weighted composition operators, *Thai J. Maths.*, **11(1)**(2013), 131–142.

[4] R.G. Douglas, On Majoriation, Factorization and Range Inclusion of Operators on Hilbert Spaces, *Proc. Amer. Math. Soc.*, **173**(1966), 413–415.

[5] P.R. Halmos, *A Hilbert space problem book*, Princeton-Toronto-London, (1967).

[6] I.H. Kim, On (p,k)- quasihyponormal operators, *Math. Ineq. and Appl.*, **7**(2004), 629–638.

[7] C.S. Kubrusly and B.P. Duggal, On posinormal operators, *Advances in Math. Sc. and Appl.*, **17**(2007), 131–148.

[8] A. Lambert, Localising sets for sigma-algebras and related point transformations, *Proc. Royal Soc. Edinburgh, Series A*, **118**(1991), 111–118.

[9] M.Y. Lee and S.H. Lee, On (p,k)-Quasiposinormal Operators, *Jr. Appl. Math and Computing*, **19(1)**(2005), 573–578.

[10] S.M. Patel, On Classes of Non-Hyponormal Operators, *Math. Nachr.*, **73**(1975), 147–150.

[11] C.R. Putnam, On Normal Operators in Hilbert space, *Amer. J. Math.*, **73**(1951), 357–362.

[12] M. Radjabalipour, On majorization and normality of operators, *Proc. Amer. Math. Soc.*, **2**(1977), 105–110.

[13] H.C. Rhaly, Posinormal Operators, *Jr. Math. Soc. Japan*, **46**(1994), 587–605.

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