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O-homomorphisms of Almost Distributive Lattices

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Abstract: The concept of O-homomorphisms is introduced in an Almost Distributive Lattice (ADL). A sufficient condition for a homomorphism to become an O-homomorphism is derived. Finally, it is proved that the image and the inverse image of an O-ideal of an ADL under an O-homomorphism are again O-ideals.

Keywords: Almost Distributive Lattice(ADL), Filter, O-homomorphism, O-ideal, Contraction, Kernel of a homomorphism

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Introduction

Lattice theory plays a vital role in information theory [3], information retrieval [2] and cryptanalysis. The concept of homomorphisms was introduced in an Almost Distributive Lattice(ADL) by U.M. Swamy and G.C. Rao [9]. The concept of O-ideals in ADLs is introduced by M. Sambasiva Rao and G.C. Rao [8] and proved that each O-ideal is an intersection of all minimal prime ideals. In general the image and the inverse image of an ideal of an ADL L under an onto homomorphism are again ideals, but it is not true in the case of O-ideals. In this paper, the concept of an O-homomorphism is introduced in ADLs and obtained some properties of these homomorphisms. A sufficient condition is derived for a homomorphism of an ADL to become an O-homomorphism. It is then proved

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that the image and the inverse image of an O-ideal under an O-homomorphism are again O-ideals. Finally, we prove that the kernel of a homomorphism is an O-ideal.

1 Preliminaries

In this section, we recall certain definitions and important results mostly from [4], [5], [6], [7], [9] and [10] those will be required in the text of the paper.

Definition 1.1. [9] An Almost Distributive Lattice(ADL)with zero is an algebra $(L, \vee, \wedge, 0)$ of type (2,2,0) satisfies the following properties:

1 $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
2 $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3 $(x \vee y) \wedge y = y$
4 $(x \vee y) \wedge x = x$
5 $x \vee (x \wedge y) = x$
6 $0 \wedge x = 0$ for any $x, y, z \in L$

Let X be a non-empty set and $x_0 \in X$ a fixed element. Then for any $x, y \in X$, define $x \vee y = y$ for $x = x_0$, otherwise $x \vee y = x$. Also $x \wedge y = x_0$ for $x = x_0$, otherwise $x \wedge y = y$. Then clearly $(X, \vee, \wedge, x_0)$ is an ADL with x_0 as zero element. This ADL is called a discrete ADL.

If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b \in L$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on L. Throughout this paper, L stands for an ADL and by an ADL we mean the ADL $(L, \vee, \wedge, 0)$.

Theorem 1.2. *[9] For any $a, b, c \in L$, we have the following.*

1 $a \vee b = a \Leftrightarrow a \wedge b = b$
2 $a \vee b = b \Leftrightarrow a \wedge b = a$
3 $a \wedge b = b \wedge a$ *whenever* $a \leq b$
4 $\wedge$ *is associative in* L
5 $a \wedge b \wedge c = b \wedge a \wedge c$
6 $(a \vee b) \wedge c = (b \vee a) \wedge c$
7 $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
8 $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
9 $a \wedge (a \vee b) = a, (a \wedge b) \vee b = b,$ *and* $a \vee (b \wedge a) = a$

10 $a \leq a \vee b$ *and* $a \wedge b \leq b$
11 $a \wedge a = a$ *and* $a \vee a = a$
12 $0 \vee a = a$ *and* $a \wedge 0 = 0$.

Definition 1.3. [9] A non-empty subset I of L is called an ideal (filter) of L if $a \vee b \in I (a \wedge b \in I)$ and $a \wedge x \in I (x \vee a \in I)$ whenever $a, b \in I$ and $x \in L$.

If I is an ideal of L and $a, b \in L$, then $a \wedge b \in I \Leftrightarrow b \wedge a \in I$. An ideal I of an ADL L is called proper if $I \neq L$. The set $\mathcal{I}(L)$ of all ideals of L is a complete distributive lattice with the least element $\{0\}$ and the greatest element L under set inclusion in which, for any $I, J \in \mathcal{I}(L)$, $I \cap J$ is the infemum of I, J and the supremum is given by $I \vee J = \{ i \vee j \mid i \in I, j \in J \}$. For any $a \in L$, $(a] = \{ a \wedge x \mid x \in L \}$ is the principal ideal generated by a. The set $\mathcal{PI}(L)$ of all principal ideals of L is a sublattice of the ideal lattice $\mathcal{I}(L)$.

Similarly, the set $\mathcal{F}(L)$ of all filters of an ADL L is also a distributive lattice in which, for any F, G of $\mathcal{F}(L)$, $F \cap G$ is the infemum of F, G and the supremum is $F \vee G = \{ f \wedge g \mid f \in F,\ g \in G \}$. For any $a \in L$, $[a) = \{ x \vee a \mid x \in L \}$ is the principal filter generated by a. The set $\mathcal{PF}(L)$ of all principal filters of L is a sublattice of the filter lattice $\mathcal{F}(L)$.

For any subset A of an ADL L, the set $A^* = \{ x \in L \mid a \wedge x = 0 \text{ for all } a \in A \}$ is an ideal of L. We write $(a]^*$ for $\{a\}^*$ and this is called an annulet of L[4]. Clearly $(0]^* = L$ and $L^* = (0]$.

Definition 1.4. [5] An ideal I of L is called an annihilator ideal if $I^{**} = I$.

Lemma 1.5. *[5] For any two ideals I, J of L, we have the following:*

1) *If* $I \subseteq J$, *then* $J^* \subseteq I^*$
2) $I^{**} \cap J^{**} = (I \cap J)^{**}$
3) $(I \vee J)^* = I^* \cap J^*$.

Definition 1.6. [8] An ideal I of L is called an O-ideal if $I = O(F) = \bigcup_{x \in F} (x]^*$ for some filter F of L.

Definition 1.7. [9] Let L and L' be two ADLs with zero elements 0 and $0'$ respectively. Then a mapping $f : L \to L'$ is called a homomorphism if it satisfies the following :

(1) $f(a \vee b) = f(a) \vee f(b)$

(2) $f(a \wedge b) = f(a) \wedge f(b)$
(3) $f(0) = 0'$.

The kernel of the homomorphism f is defined by $Ker f = \{x \in L \mid f(x) = 0'\}$. Then clearly $Ker\ f$ is an ideal in L.

2 O-homomorphisms

In this section, the concept of O-homomorphisms is introduced in an almost distributive lattices (ADL) and a sufficient condition is derived for an ADL homomorphism to become an O-homomorphism. Throughout this section, L and L' stand for two ADLs with zeroes 0 and $0'$ respectively.

We begin this section with the following lemma which we need later.

Lemma 2.1. *Let L and L' be two ADLs. If $f : L \to L'$ is a homomorphism, then we have the following :*

(1) For any filter F of L', $f^{-1}(F)$ is a filter of L provided $f^{-1}(F) \neq \emptyset$.
(2) If f is onto, then for any filter G of L, $f(G)$ is a filter of L'.

Proof. (1). Let F be a filter of L' such that $f^{-1}(F) \neq \emptyset$. Let $a, b \in f^{-1}(F)$. Then $f(a), f(b) \in F$. Since F is a filter of L', we get that $f(a \wedge b) = f(a) \wedge f(b) \in F$. Hence $a \wedge b \in f^{-1}(F)$. Again, let $x \in f^{-1}(F)$ and $r \in L$. Then $f(x) \in F$. Now $f(r \vee x) = f(r) \vee f(x) \in F$. Hence $r \vee x \in f^{-1}(F)$. Thus $f^{-1}(F)$ is a filter of L.
(2). Since G is non-empty, we get that $f(G)$ is non-empty. Let $f(a), f(b) \in f(G)$ where $a, b \in G$. Then $f(a) \wedge f(b) = f(a \wedge b) \in f(G)$. Again, let $f(a) \in f(G)$ and $x \in L'$. Since f is on-to, there exists $y \in L$ such that $f(y) = x$. Now $x \vee f(a) = f(y) \vee f(a) = f(y \vee a) \in f(G)$. Therefore $f(G)$ is a filter of L'. $\square$

Lemma 2.2. *Let L and L' be two ADLs. If $f : L \to L'$ is a homomorphism, then for any filter F of L, we have:*

$$f\big[O(F)\big] \subseteq O\big[f(F)\big].$$

Proof. Let $x \in f\big[O(F)\big]$. Then $x = f(a)$ for some $a \in O(F)$. Now $a \in O(F)$ implies that $a \wedge s = 0$ for some $s \in F$. Then $x \wedge f(s) = f(a) \wedge f(s) = f(a \wedge s) = f(0) = 0'$. Hence $x \in O\big[f(F)\big]$. Therefore $f\big[O(F)\big] \subseteq O\big[f(F)\big]$. $\square$

In general, for any filter F of an ADL L, $f\big[O(F)\big] = O\big[f(F)\big]$ is not true. For consider the following example:

Example 2.3. Let $L = \{0, a, b, c\}$ be a discrete ADL. Define a mapping $f : L \to L$ by $f(x) = 0$ for all $x \in L$. Then clearly f is a homomorphism on L. Now for any filter F of L, $O(F) = \bigcup_{x \in F}(x]^* = \{0\}$. Hence $f\big[O(F)\big] = f\big(\{0\}\big) = \{0\}$. Also $f(F) = \{0\}$. Hence $O\big[f(F)\big] = (0]^* = L$. Therefore $f\big[O(F)\big] \neq O\big[f(F)\big]$.

We now introduce the concept of O-homomorphisms in the following.

Definition 2.4. Let L and L' be two ADLs. Then a homomorphism $f : L \to L'$ is called an O-homomorphism if

$$f\big[O(F)\big] = O\big[f(F)\big]$$

for any filter F of L. An onto O-homomorphism is called an O-epimorphism.

Example 2.5. Let $L_1 = \{0, b_1, b_2\}$ and $L_2 = \{0, a\}$ be two discrete ADLs. Define a mapping $f : L_1 \longrightarrow L_2$ by $f(0) = 0, f(b_1) = f(b_2) = a$. Then clearly f is a homomorphism from L_1 onto L_2. Clearly $F = \{b_1, b_2\}$ is the only filter of L_1. Now $O(F) = (b_1]^* \bigcup\ (b_2]^* = \{0\}$ and $f(F) = \{a\}$. Hence $f\big[O(F)\big] = O\big[f(F)\big]$. Therefore f is an O-homomorphism of L_1.

If f is a ring epimorphism, then it is an isomorphism if and only if $Ker\ f = \{0\}$. But it need not be true in the case of ADL-homomorphisms. It may be seen from the above Example 2.5. Clearly f is onto and $Ker\ f = \{0\}$. But f is not one-one. However, we have the following:

Theorem 2.6. *Let L and L' be two ADLs and $f : L \longrightarrow L'$ be an epimorphism. If $Ker\ f = \{0\}$, then f is an O-homomorphism.*

Proof. Assume that f is onto and $Ker\ f = \{0\}$. Let F be a filter of L. We have always $f\big[O(F)\big] \subseteq O\big[f(F)\big]$. Let $x \in O\big[f(F)\big] \subseteq L'$. Since f is onto, there exists $y \in L$ such that $f(y) = x$. Now

$$\begin{aligned} f(y) \in O\big[f(F)\big] &\Rightarrow f(y) \wedge f(s) = 0' \text{ for some } s \in F \\ &\Rightarrow f(y \wedge s) = 0' \\ &\Rightarrow y \wedge s \in Ker\ f = \{0\} \\ &\Rightarrow y \in O(F) \\ &\Rightarrow x = f(y) \in f\big[O(F)\big]. \end{aligned}$$

Hence $O\big[f(F)\big] \subseteq f\big[O(F)\big]$. Therefore $f\big[O(F)\big] = O\big[f(F)\big]$. □

Theorem 2.7. *Let L and L' be two ADLs and $f : L \longrightarrow L'$ an epimorphism such that $Ker\ f = \{0\}$. Then $O(F) = O(G)$ if and only if $O\big[f(F)\big] = O\big[f(G)\big]$ for any two filters F, G of L.*

Proof. Assume that f is an epimorphism and Ker $f = \{0\}$. Then by above theorem, f is an O-homomorphism. Let F, G be two filters of L. Suppose $O(F) = O(G)$. Then $f\big[O(F)\big] = f\big[O(G)\big]$. Hence $O\big[f(F)\big] = O\big[f(G)\big]$. Conversely, assume that $O\big[f(F)\big] = O\big[f(G)\big]$. Now

$$\begin{aligned} t \in O(F) &\Rightarrow f(t) \in f\big[O(F)\big] \\ &\Rightarrow f(t) \in O\big[f(F)\big] \\ &\Rightarrow f(t) \in O\big[f(G)\big] \\ &\Rightarrow f(t) \wedge f(s) = 0' \quad \text{for some } s \in G \\ &\Rightarrow f(t \wedge s) = 0' \\ &\Rightarrow t \wedge s \in Ker\ f = \{0\} \\ &\Rightarrow t \wedge s = 0 \text{ and } s \in G \\ &\Rightarrow t \in O(G). \end{aligned}$$

Therefore $O(F) \subseteq O(G)$. Similarly, we can have $O(G) \subseteq O(F)$. Therefore $O(F) = O(G)$. □

Theorem 2.8. *Let L and L' be two ADLs and $f : L \longrightarrow L'$ an O-homomorphism. Then $f(K)$ is an O-ideal of L', for any O-ideal K of L.*

Proof. Let K be an O-ideal of L. Then $K = O(F)$ for some filter F of L. Then by Lemma 2.1, $f(F)$ is a filter of L'. Since f is an O-homomorphism, we have that $f(K) = f\big[O(F)\big] = O\big[f(F)\big]$. Therefore $f(K)$ is an O-ideal of L. □

We now define the contraction of an ideal in an ADL in the following:

Definition 2.9. Let L and L' be two ADLs and $f : L \longrightarrow L'$ a homomorphism. For any ideal J of L', $f^{-1}(J)$ is called the contraction of J with respect to f.

If $f : L \longrightarrow L'$ is a homomorphism of ADLs, then we know that the contraction of every ideal of L' is an ideal in L. But in the case of O-ideals it may not be true. For consider the following example:

Example 2.10. Let $L=\{0,a,b,c\}$ and define $\vee$ and $\wedge$ on L as follows:

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	a	b	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

Then clearly $(L,\vee,\wedge,0)$ is an ADL with 0.
Let $L'=A\times B$ where $A=\{0,a'\}$ and $B=\{0',b_1,b_2\}$ be two discrete ADLs. Then $(L',\vee,\wedge,\bar{0})$ is an ADL with respect to the point wise operations and $\bar{0}=(0,0')$. Now define a mapping $f:L\longrightarrow L'$ as follows:

$$f(0)=\bar{0}, f(a)=(a',b_1), f(b)=(a',b_2) \text{ and } f(c)=(a',0).$$

Then clearly f is a homomorphism from L into L'. Now consider the ideal $J=\{(0,0'),(a',0')\}$ and the filter $F=\{(a',b_1),(a',b_2),(0,b_1),(0,b_2)\}$ of L'. Clearly $J=O(F)$. Hence J is an O-ideal of L'. But $f^{-1}(J)=\{0,c\}$ is an ideal of L which is not an O-ideal, because $c\in f^{-1}(J)$ and $(c]^*=(0]$.

However, in the following, we derive a sufficient condition for the contraction of every O-ideal is again an O-ideal.

Theorem 2.11. *Let L and L' be two ADLs and $f:L\longrightarrow L'$ an epimorphism such that $Ker\ f=\{0\}$. If every filter of L' contracts to a filter of L, then every O-ideal of L' contracts to an O-ideal of L.*

Proof. Let J be an O-ideal of L'. Then $J=O(G)$ for some filter G of L'. By hypothesis, $f^{-1}(G)$ is a filter of L. We now show that $f^{-1}\big[O(G)\big]=O\big[f^{-1}(G)\big]$. Let $x\in O\big[f^{-1}(G)\big]$. Then $x\wedge s=0$ for some $s\in f^{-1}(G)$. Now

$$\begin{aligned} x\wedge s=0 &\Rightarrow f(x)\wedge f(s)=f(0)=0' \text{ and } f(s)\in G\\ &\Rightarrow f(x)\in O(G)\\ &\Rightarrow x\in f^{-1}\big[O(G)\big]. \end{aligned}$$

Hence $O\big[f^{-1}(G)\big] \subseteq f^{-1}\big[O(G)\big]$. Conversely, let $x \in f^{-1}\big[O(G)\big]$. Then we get

$$\begin{aligned} f(x) \in O(G) &\Rightarrow f(x) \wedge f(t) = 0' \text{ for some } f(t) \in G \\ &\Rightarrow f(x \wedge t) = 0' \\ &\Rightarrow x \wedge t = Ker\ f = \{0\} \\ &\Rightarrow x \wedge t = 0 \text{ and } t \in f^{-1}(G) \\ &\Rightarrow x \in O\big[f^{-1}(G)\big]. \end{aligned}$$

Hence $f^{-1}\big[O(G)\big] \subseteq O\big[f^{-1}(G)\big]$. Therefore $f^{-1}\big[O(G)\big] = O\big[f^{-1}(G)\big]$. □

Theorem 2.12. *Let $f : L \longrightarrow L'$ be a homomorphism such that each O-ideal of L' contracts to an O-ideal of L. If L' has a dense element, then $Ker\ f$ is an O -ideal in L.*

Proof. Since L' has a dense element, $\{0'\}$ is an O-ideal in L'. Hence $Ker\ f = f^{-1}\big(\{0'\}\big)$ is an O-ideal in L. □

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