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O-ideals in Almost Distributive Lattices

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Abstract: Concept of O-ideals is introduced in an Almost Distributive Lattice and these O-ideals are characterized in terms of minimal prime ideals. A necessary and sufficient condition for the zero ideal of an ADL to become an O-ideal is derived. $\star$-ADLs are characterized in terms of their O-ideals and α-ideals. Finally, a set of equivalent conditions are established for every O-ideal of an ADL to become a principal ideal.

Keywords: Almost Distributive Lattice(ADL), O-ideal, α-ideal, $\star$-ADL

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Introduction

Lattice theory plays a vital role in information theory [4], information retrieval[2] and cryptanalysis. W.H. Cornish [3] introduced the concept of O-ideals and studied their properties in a distributive lattice. The concept of an Almost Distributed Lattice(ADL) was introduced by U.M. Swamy and G.C. Rao [9]. Later, U.M. Swamy, G.C. Rao and G. Nanaji Rao [10] introduced a general class called $\star$-ADLs. In [7], the concept of α-ideals was introduced in ADLs and the properties of these ideals were studied. In this paper, the concept of O-ideals is introduced in an ADL analogous to that in a distributive lattice. In [5] and [7], authors observed that the zero ideal $\{0\}$ is an annihilator ideal as well as an α-ideal. In

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this paper, it is observed that $\{0\}$ need not be an O-ideal and hence a necessary and sufficient condition is derived for the ideal $\{0\}$ to become an O-ideal. It is also proved that each O-ideal of an ADL is the intersection of all minimal prime ideals containing it. Finally, it is observed that every O-ideal of an ADL is an α-ideal, but the converse is not true in general. However, some equivalent conditions are derived for every α-ideal of an ADL to become an O-ideal, which leads to a characterization of $\star$-ADLs.

1 Preliminaries

In this section, we recall certain definitions and important results mostly from [5], [6], [7], [8], [9] and [10] those will be required in the text of the paper.

Definition 1.1. [9] An Almost Distributive Lattice(ADL)with zero is an algebra $(L, \vee, \wedge, 0)$ of type (2,2,0) satisfies the following properties:

1. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3. $(x \vee y) \wedge y = y$
4. $(x \vee y) \wedge x = x$
5. $x \vee (x \wedge y) = x$
6. $0 \wedge x = 0$ for any $x, y, z \in L$

Let X be a non-empty set and $x_0 \in X$ a fixed element. Then for any $x, y \in X$, define $x \vee y = y$ for $x = x_0$, otherwise $x \vee y = x$. Also $x \wedge y = x_0$ for $x = x_0$, otherwise $x \wedge y = y$. Then clearly $(X, \vee, \wedge, x_0)$ is an ADL with x_0 as zero element and is called a discrete ADL. If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b \in L$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on L. Throughout this paper, L stands for an ADL and by an ADL we mean the ADL $(L, \vee, \wedge, 0)$.

Theorem 1.2. *[9] For any $a, b, c \in L$, we have the following.*

1. $a \vee b = a \Leftrightarrow a \wedge b = b$
2. $a \vee b = b \Leftrightarrow a \wedge b = a$
3. $a \wedge b = b \wedge a$ *whenever* $a \leq b$
4. $\wedge$ *is associative in* L
5. $a \wedge b \wedge c = b \wedge a \wedge c$

6. $(a \vee b) \wedge c = (b \vee a) \wedge c$
7. $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
8. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
9. $a \wedge (a \vee b) = a, (a \wedge b) \vee b = b,$ *and* $a \vee (b \wedge a) = a$
10. $a \leq a \vee b$ *and* $a \wedge b \leq b$
11. $a \wedge a = a$ *and* $a \vee a = a$
12. $0 \vee a = a$ *and* $a \wedge 0 = 0$.

A non-empty subset I of L is called an ideal(filter)[9] of L if $a \vee b \in I (a \wedge b \in I)$ and $a \wedge x \in I (x \vee a \in I)$ whenever $a, b \in I$ and $x \in L$. If I is an ideal of L and $a, b \in L$, then $a \wedge b \in I \Leftrightarrow b \wedge a \in I$. The set $\mathcal{I}(L)$ of all ideals of L is a complete distributive lattice with the least element $\{0\}$ and the greatest element L under set inclusion in which, for any $I, J \in \mathcal{I}(L)$, $I \cap J$ is the infemum of I, J and the supremum is given by $I \vee J = \{ i \vee j \mid i \in I, j \in J \}$. Similarly, the set $\mathcal{F}(L)$ of all filters of L is also a distributive lattice in which, for any F, G of $\mathcal{F}(L)$, $F \cap G$ is the infemum of F, G and the supremum is $F \vee G = \{ f \wedge g \mid f \in F, \ g \in G \}$. An ideal I of an ADL L is called proper if $I \neq L$. For any $a \in L$, $(a] = \{ a \wedge x \mid x \in L \}$ is the principal ideal generated by a. Similarly, for any $a \in L$, $[a) = \{ x \vee a \mid x \in L \}$ is the principal filter generated by a. The set $\mathcal{PI}(L)$ of all principal ideals of L is a sublattice of $\mathcal{I}(L)$. A proper ideal(filter) P of L is said to be prime if for any $x, y \in L$, $x \wedge y \in P (x \vee y \in P) \Rightarrow x \in P$ or $y \in P$. It is clear that a subset P of L is a prime ideal if and only if $L - P$ is a prime filter.

A prime ideal P of L is called a minimal prime ideal belonging to an ideal I of L if $I \subseteq P$ and there is no prime ideal Q of L such that $I \subseteq Q \subset P$[8]. A prime ideal belonging to zero ideal is called a minimal prime ideal.

Theorem 1.3. *[8] (i). A prime ideal P of L is a minimal prime ideal belonging to an ideal I if and only if to each $x \in P$ there exists $y \notin P$ such that $x \wedge y \in I$.*
(ii). A prime ideal P of an ADL L is a minimal prime ideal if and only if to each $x \in P$ there exists $y \notin P$ such that $x \wedge y = 0$.
(iii).[7] The intersection of all minimal prime ideals is $\{0\}$.

For any $A \subseteq L$, $A^* = \{ x \in L \mid a \wedge x = 0 \text{ for all } a \in A \}$ is an ideal of L. We write $(a]^*$ for $\{a\}^*$ and this is called an annulet of L[4]. Clearly $(0]^* = L$ and $L^* = (0]$. An ideal I of L is called an annihilator ideal if $I^{**} = I$[5].

Lemma 1.4. *[5] For any two ideals I, J of L, we have the following:*

1). If $I \subseteq J$*, then* $J^* \subseteq I^*$
2). $I^{**} \cap J^{**} = (I \cap J)^{**}$
3). $(I \vee J)^* = I^* \cap J^*$

An ideal I of an ADL is called an α-ideal if $(x]^{**} \subseteq I$, for each $x \in I$[7]. For any ideal I of L, the set $I^e = \{ x \in L \mid (a]^* \subseteq (x]^* \text{ for some } a \in I \}$ is the smallest α-ideal such that $I \subseteq I^e$. Clearly the intersection of α-ideals is again an α-ideal and every annihilator ideal is an α-ideal. An element $x \in L$ is called dense[11] if $(x]^* = (0]$. The set D of all dense elements of L forms a filter, whenever D is nonempty. An ADL L is called a $\star$-ADL, if to each $x \in L$, $(x]^{**} = (x']^*$ for some $x' \in L$[10]. Every $\star$-ADL possesses a dense element.

2 Properties of O-ideals

In this section, the concept of O-ideals is introduced in an ADL and some properties of these O-ideals are studied. A set of equivalent conditions are established for every O-ideal of an ADL to become a principal ideal.

Definition 2.1. For any filter F of an ADL L, define

$$O(F) \; = \; \{ x \in L \mid x \wedge f = 0 \text{ for some } f \in F \}$$

We first observe some elementary properties of $O(F)$ in the following lemmas.

Lemma 2.2. *For any filter* F *of an ADL* L*,* $O(F)$ *is an ideal in* L*.*

Proof. Clearly $0 \in O(F)$. Let $a, b \in O(F)$. Then $a \wedge f = b \wedge g = 0$ for some $f, g \in F$. Now $(a \vee b) \wedge (f \wedge g) = (a \wedge f \wedge g) \vee (b \wedge f \wedge g) = (0 \wedge g) \vee (f \wedge b \wedge g) = 0 \vee (f \wedge 0) = 0$. Hence $a \vee b \in O(F)$. Again, let $a \in O(F)$ and $x \in L$. Then $a \wedge f = 0$ for some $f \in F$. Now $(a \wedge x) \wedge f = x \wedge a \wedge f = x \wedge 0 = 0$. So $a \wedge x \in O(F)$. Thus $O(F)$ is an ideal in L. □

The following lemma is a routine verification.

Lemma 2.3. *For any two filters* F, G *of an ADL* L*, we have the following:*

(1). $O(F) \; = \; \bigcup_{x \in F} (x]^*$
(2). $F \subseteq G$ *implies* $O(F) \subseteq O(G)$
(3). $O(F \cap G) \; = \; O(F) \cap O(G)$
(4). $O(F) \vee O(G) \subseteq O(F \vee G)$.

In the above lemma, property (3) is true even for any family of filters of L. That is, if $\{F_\alpha | \alpha \in \Delta\}$ is a family of filters of L, then

$$O(\bigcap_{\alpha \in \Delta} F_\alpha) = \bigcap_{\alpha \in \Delta} O(F_\alpha)$$

Lemma 2.4. *Let L be an ADL with dense elements. Then for any filter F of L,*

$$O(F) = L \quad \textit{if and only if} \quad F = L.$$

Proof. Let d be a maximal element of L. Assume that $F = L$. Then $O(F) = O(L) = \bigcup_{x \in L} (x]^* = L$ (since $0 \in L$). Conversely, assume that $O(F) = L$. Then $d \in O(F)$. Hence $d \wedge f = 0$ for some $f \in F$. Thus $f = 0$. Therefore $F = L$. $\square$

The concept of O-ideals is now introduced in an ADL L.

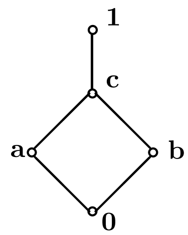
Definition 2.5. An ideal I of an ADL L is called an O-ideal if and only if $I = O(F)$ for some filter F of L.

Example 2.6. Let $A = \{0, a\}$ and $B = \{0, b_1, b_2\}$ be two discrete ADLs. Write $L = A \times B = \{(0,0), (0,b_1), (0,b_2), (a,0), (a,b_1), (a,b_2)\}$. Then $(L, \vee, \wedge, 0')$ is an ADL where $0' = (0,0)$, under point-wise operations. Consider the ideal $I = \{(0,0), (0,b_1), (0,b_2)\}$ and filter $F = \{(a,0), (a,b_1), (a,b_2)\}$. Now

$$\begin{aligned} O(F) = \bigcup_{x \in F} (x]^* &= ((a,0)]^* \cup ((a,b_1)]^* \cup ((a,b_2)]^* \\ &= \{(0,0), (0,b_1), (0,b_2)\} \cup \{(0,0)\} \cup \{0,0)\} \\ &= \{(0,0), (0,b_1), (0,b_2)\} \end{aligned}$$

Therefore $I = O(F)$ and hence I is an O-ideal of L.

Example 2.7. Let $L = \{0, a, b, c, 1\}$ be a distributive lattice whose Hasse diagram is given in the following figure:



Consider $I = \{0, a\}$ and $J = \{0, a, b, c\}$.
Clearly I and J are ideals of L.
Also $F = \{b, c, 1\}$ is a filter of L.
Clearly $O(F) = \bigcup_{x \in F} (x]^* = \{0, a\} = I$.
But J is not an O-ideal of L.

In the above example, the reason for the ideal J of L not being an O-ideal can be observed as a simple property of O-ideals of L, in the following lemma.

Lemma 2.8. *A proper O-ideal of an ADL L contains no dense elements.*

Proof. Let I be an O-ideal and d a dense element of L. Since I is an O-ideal, we get that $I = O(F)$ for some filter F of L. Suppose $d \in I = O(F)$. Then $d \wedge f = 0$ for some $0 \neq f \in F$. Thus $f \in (d]^*$. Which is a contradiction to that d is a dense element of L. □

It was already observed in [5, 7] that the zero ideal $\{0\}$ is an annihilator ideal as well as an α-ideal in an ADL L. But, in general, it is not an O-ideal in L. It may be observed in the following example.

Example 2.9. Let $L = \{0, a, b\}$ be a discrete ADL and X is any infinite set. Take $L' = \{ f \in L^X \mid \text{ support } f \text{ is finite } \}$ where support $f = \{x \in X \mid f(x) \neq 0\}$. Define $\vee$ and $\wedge$ on L' point-wise. Then $(L', \vee, \wedge, f_0)$ is an ADL with f_0 as zero where $f_0(x) = 0$ for all $x \in X$. We first observe that L' has no dense elements. Suppose $f \in L'$.
Choose $x \in X-$ Support f and define $g : X \longrightarrow L$ by

$$g(t) = \begin{cases} a & \text{if } t = x \\ 0 & \text{if } t \neq x \end{cases}$$

Then clearly support $g = \{x\}$ and hence $f_0 \neq g \in L'$. Now, $(f \wedge g)(t) = f(t) \wedge g(t) = 0$ for all $t \in X$. Hence $g \in (f]^*$. Thus $(f]^* \neq (f_0]$. Therefore L' has no dense elements. Suppose the zero ideal $(f_0]$ of L' is an O-ideal. Then

$$\begin{aligned} (f_0] &= O(F) \quad \text{for some filter } F \text{ of } L' \\ &= \bigcup_{x \in F} (f]^* \end{aligned}$$

Hence we get $(f]^* = (f_0]$ for all $f \in F$. Thus we have obtained that each $f \in F$ is a dense element. Which is a contradiction to the fact that L' has no dense elements. Therefore the zero ideal $(f_0]$ is not an O-ideal of L'.

However, a necessary and sufficient condition for the zero ideal of an ADL L to become an O-ideal, is derived in the following theorem.

Theorem 2.10. *The zero ideal is an O-ideal if and only if L has a dense element.*

Proof. Assume that $\{0\}$ is an O-ideal of L. Then $\{0\} = O(F) = \bigcup_{x \in F} (x]^*$ for some filter F of L. Hence $(x]^* = (0]$ for each $x \in F$. Thus L has a dense element, since $F \neq \emptyset$. Conversely, assume that L has a dense element. Then the set D of all dense elements of L is a filter of L. Also $O(D) = \bigcup_{x \in D}(x]^* = \{0\}$. Therefore $\{0\}$ is an O-ideal of L. □

Let us denote the set of all O-ideals of an ADL L by $\mathcal{I}_o(L)$. In lemma 2.3, it is observed that $O(F) \vee O(G) \subseteq O(F \vee G)$ for any two filters F, G of L. In general, equality does not hold in an ADL. It may be seen in example 2.7. Consider the two filters $F = \{b, c, 1\}$ and $G = \{a, c, 1\}$. Then $O(F) = \{0, a\}$, $O(G) = \{0, b\}$ and $O(F) \vee O(G) = \{0, a, b, c\}$. But $O(F \vee G) = O(L) = L$. Hence $O(F) \vee O(G) \neq O(F \vee G)$. Thus $I_o(L)$ is not a sublattice of the distributive lattice $\mathcal{I}(L)$ of all ideals of L. However, in the following, it is proved that $I_o(L)$ forms a bounded and complete distributive lattice on its own.

Theorem 2.11. *For any ADL L, $\mathcal{I}_o(L)$ is a distributive lattice on its own.*

Proof. For any two filters F, G of L, define binary operations $\cap$ and $\sqcup$ as follows:

$$O(F) \cap O(G) \ = \ O(F \cap G) \quad \text{and} \quad O(F) \sqcup O(G) \ = \ O(F \vee G)$$

By lemma 2.3(3), $O(F \cap G)$ is the infemum of $O(F)$ and $O(G)$ in $\mathcal{I}_o(L)$. Also $O(F) \sqcup O(G)$ is an O-ideal of L. Clearly $O(F), O(G) \subseteq O(F \vee G) = O(F) \sqcup O(G)$. Let $O(H)$ be an O-ideal of L such that $O(F) \subseteq O(H)$ and $O(G) \subseteq O(H)$, where H is a filter of L. Now

$$\begin{aligned} x \in O(F \vee G) &\Rightarrow x \wedge a = 0 \quad \text{for some } a \in F \vee G \\ &\Rightarrow x \wedge f \wedge g = 0 \quad \text{for some } f \in F \text{ and } g \in G. \\ &\Rightarrow x \wedge f \in O(G) \subseteq O(H) \\ &\Rightarrow x \wedge f \wedge h_1 = 0 \quad \text{for some } h_1 \in H \\ &\Rightarrow x \wedge h_1 \in O(F) \subseteq O(H) \\ &\Rightarrow x \wedge h_1 \wedge h_2 = 0 \quad \text{for some } h_2 \in H \\ &\Rightarrow x \in O(H) \quad (\text{since } h_1 \wedge h_2 \in H). \end{aligned}$$

Therefore $O(F \vee G)$ is the supremum of $O(F), O(G)$ in $\mathcal{I}_o(L)$. Now, it can be easily verified that $(I_o(L), \cap, \sqcup)$ is a distributive lattice. □

If L has dense elements, then we have the following:

Theorem 2.12. *Let L be an ADL with dense elements. Then the lattice $\mathcal{I}_o(L)$ is bounded and complete.*

Proof. Let F, G be two filters of L. Then $O(F), O(G) \in \mathcal{I}_o(L)$. Define $O(F) \leq O(G) \Leftrightarrow O(F) \subseteq O(G)$. Clearly $(\mathcal{I}_o(L), \leq)$ is a partially ordered set. Clearly $\{0\}$

and L are the O-ideals in L and they are the bounds for $\mathcal{I}_o(L)$. Therefore by the extension of lemma 2.3(3), we get that $\mathcal{I}_o(L)$ is a complete lattice. □

In view of the above two theorems, we can conclude that $(I_o(L), \cap, \sqcup, \{0\}, L)$ is a bounded and complete distributive lattice. Finally, in this section, we give a set of equivalent conditions for every O-ideal of L to become a principal ideal.

We first prove the following lemma.

Lemma 2.13. *If each $(x]^*$ of an ADL L is a principal ideal, then every prime α-ideal of L is a minimal prime ideal.*

Proof. Let P be a prime α-ideal of L. Then clearly $P \cap D \neq \emptyset$. Let $x \in P$. Then by hypothesis, $(x]^* = (y]$ for some $y \in L$. Hence $x \wedge y = 0$. Now $(x \vee y]^* = (x]^* \cap (y]^* = (x]^* \cap (x]^{**} = (0]$. Hence $x \vee y \notin P$. Thus $y \notin P$. Therefore P is a minimal prime ideal. □

Theorem 2.14. *The following conditions are equivalent in an ADL L.*

(1). Every α-ideal is a principal ideal
(2). Every O-ideal is a principal ideal
(3). Each $(x]^*$ is a principal ideal and every minimal prime ideal is non-dense
(4). Every prime α-ideal is a principal ideal.

Proof. $(1) \Rightarrow (2)$: Since every O-ideal is an α-ideal, it is clear.

$(2) \Rightarrow (3)$: Assume that every O-ideal is a principal ideal. Since each $(x]^*$ is an O-ideal, it remains to prove that every minimal prime ideal is non-dense. Let P be a minimal prime ideal of L. Then $L-P$ is a filter. Since P is minimal, we get $P = O(L-P)$. Hence P is an O-ideal. Thus $P = (a]$ for some $a \in L$. Suppose $P^* = (a]^* = (0]$. Then $a \in P$ is a dense element. Which is a contradiction to that a proper O-ideal does not contain a dense element. Therefore P is non-dense.

$(3) \Rightarrow (4)$: Assume the condition (3). Let P be a prime α-ideal of L. Then by above lemma, P is a minimal prime ideal. By hypothesis, P is non-dense. Hence $P = (x]^*$ for some $0 \neq x \in L$. Again by hypothesis $P = (x]^*$ is a principal ideal.

$(4) \Rightarrow (1)$: Assume condition (4). Let I be an α-ideal. Suppose that I is not principal. Consider $\Sigma = \{\ J \mid J \text{ is an } \alpha\text{-ideal which is not a principal ideal } \}$. Then clearly $I \in \Sigma$. Let $\{J_i\}_{i\in\Delta}$ be a chain in Σ. Clearly $\bigcup J_i$ is an α-ideal in Σ. Suppose $\bigcup J_i = (a]$ for some $a \in L$. Then $a \in \bigcup J_i$ implies that $a \in J_i$ for some $i \in \Delta$. Hence $(a] \subseteq J_i$ for some $i \in \Delta$. On the other hand, $J_i \subseteq \bigcup J_i = (a]$.

Hence $J_i = (a]$ for some $i \in \Delta$. Which is a contradiction. Thus $\bigcup J_i$ is an upper bound for $\{J_i\}_{i\in\Delta}$ in Σ. Let M be a maximal element of Σ. Choose $x, y \in L$ such that $x \notin M$ and $y \notin M$. Then $M \subset M \vee (x] \subseteq \{M \vee (x]\}^e$ and $M \subset M \vee (y] \subseteq \{M \vee (y]\}^e$. By the maximality of M, we get $\{M \vee (x]\}^e = (b]$ and $\{M \vee (y]\}^e = (c]$ for some $b, c \in L$. Hence $\{M \vee (x \wedge y]\}^e = \{M \vee (x]\}^e \cap \{M \vee (y]\}^e = (b] \cap (c] = (b \wedge c]$. If $x \wedge y \in M$, then $M = M^e = (b \wedge c]$. Which is a contradiction. Hence M is a prime α-ideal which is not a principal ideal. Which is a contradiction to the hypothesis. Therefore I is a principal ideal. □

3 Characterization of O-ideals

In this section, O-ideals are first characterized in terms of minimal prime ideals. Later, the class of $\star$-ADLs are characterized in terms of O-ideals and α-ideals.

First we prove the following two important lemmas.

Lemma 3.1. *Let F be a filter of an ADL L. If P is a minimal prime ideal belonging to the ideal $O(F)$, then $P \cap F = \emptyset$.*

Proof. Let P be a minimal prime ideal belonging to the ideal $O(F)$. Suppose $x \in P \cap F$. Then $x \in P$ and $x \in F$. Since P is minimal, there exists $y \notin P$ such that $x \wedge y \in O(F)$. Then $x \wedge y \wedge f = 0$ for some $f \in F$. Hence we get $y \wedge (x \wedge f) = 0$ and $x \wedge f \in F$. Thus $y \in O(F) \subseteq P$. Which is a contradiction. □

Lemma 3.2. *Every minimal prime ideal of an ADL L belonging to an O-ideal is a minimal prime ideal in L.*

Proof. Let I be an O-ideal of L. Then we have $I = O(F)$ for some filter F of L. Let P be a minimal prime ideal belonging to $I = O(F)$. Then by above lemma, $P \cap F = \emptyset$. Let $x \in P$. Then there exists $y \notin P$ such that $x \wedge y \in O(F)$. Hence $x \wedge y \wedge f = 0$ for some $f \in F$. Thus $x \wedge (y \wedge f) = 0$ and $y \wedge f \notin P$(since $P \cap F = \emptyset$, we get that $f \notin P$ and also $y \notin P$). Hence P is a minimal prime ideal of L. □

Now, O-ideals are characterized in terms of minimal prime ideals.

Theorem 3.3. *Every O-ideal of an ADL L is the intersection of all minimal prime ideals containing it.*

Proof. Let I be an O-ideal of an ADL L. Then $I = O(F)$ for some filter F of L. Let $I_0 = \bigcap\{ P \mid P$ is a minimal prime ideal containing I $\}$. Clearly $I \subseteq I_0$. Conversely, let $a \notin I = O(F)$. Then $a \wedge t \neq 0$ for all $t \in F$. Then there exists a minimal prime ideal P such that $a \wedge t \notin P$. Hence $a \notin P$ and $t \notin P$. Since P is prime, $(t]^* \subseteq P$ for all $t \in F$. Therefore $I = O(F) \subseteq P$. Thus P is a minimal prime ideal containing I and $a \notin P$. Therefore we get $a \notin I_0$. Which yields that $I_0 \subseteq I$. Therefore $I = I_0$. □

In [5], the authors proved that the class of all α-ideals of an ADL L formed a distributive lattice on its own. In the following, it is proved that the class of all α-ideals of an ADL L properly includes the class of all O-ideals of L.

Theorem 3.4. *Every O-ideal of an ADL is an α-ideal.*

Proof. Let L be an ADL. Suppose I is an O-ideal of L. Then $I = O(F)$ for some filter F of L. Let $x \in O(F)$. Then $x \in (f]^*$ for some $f \in F$. Hence $(x]^{**} \subseteq (f]^* \subset O(F)$. Thus $(x]^{**} \subseteq I$. Therefore I is an α-ideal of L. □

But the converse of the above theorem is not true. That is, every α-ideal of an ADL L need not be an O-ideal. For, consider the distributive lattice in the example 2.7. The zero ideal $\{0\}$ of L is an α-ideal but not an O-ideal. However, in the following theorem, some equivalent conditions for every α-ideal of an ADL L to become an O-ideal are derived, which leads to a characterization of $\star$-ADLs.

Theorem 3.5. *Let L be an ADL. Then the following are equivalent:*

(1). L is a $\star$-ADL
(2). Every α-ideal is an O-ideal
(3). Every annihilator ideal is an O-ideal
(4). For $x \in L$, $(x]^{**}$ is an O-ideal.

Proof. $(1) \Rightarrow (2)$: Assume that L is a $\star$-ADL. Let I be an α-ideal of L. Consider the set $I^0 = \{ x \in L \mid (a]^* \subseteq (x]^{**}$ for some $a \in I \}$. We first prove that I^0 is a filter of L. Clearly $\emptyset \neq D \subseteq I^0$. Let $x, y \in I^0$. Then we get $(a]^* \subseteq (x]^{**}$ and $(b]^* \subseteq (y]^{**}$ for some $a, b \in I$. Now $(a \vee b]^* = (a]^* \cap (b]^* \subseteq (x]^{**} \cap (y]^{**} = (x \wedge y]^{**}$ and $a \vee b \in I$. Hence $x \wedge y \in I^0$. Again, let $x \in I^0$ and $r \in L$. Then we get that $(a]^* \subseteq (x]^{**}$ for some $a \in I$. Now $(r \vee x]^* \subseteq (x]^* \subseteq (a]^{**}$. Hence $(a]^* \subseteq (r \vee x]^{**}$ and $a \in I$. Hence $r \vee x \in I^0$. Therefore I^0 is a filter of L. We now show that $I = O(I^0)$. Let $x \in O(I^0)$. Then $x \wedge f = 0$ for some $f \in I^0$. Hence $x \in (f]^*$.

Now

$$\begin{aligned} f \in I^0 &\Rightarrow (a]^* \subseteq (f]^{**} \quad \text{for some } a \in I \\ &\Rightarrow (f]^* \subseteq (a]^{**} \subseteq I \qquad (\because I \text{ is an } \alpha\text{-ideal and } a \in I) \\ &\Rightarrow x \in I \end{aligned}$$

Therefore $O(I^0) \subseteq I$. Conversely, let $x \in I$. Since L is a $\star$-ADL, there exists $y \in L$ such that $(x]^* = (y]^{**}$. Since $x \in I$, we get that $y \in I^0$. Also $x \in (x]^{**} = (y]^*$ and $y \in I^0$. Hence $x \in O(I^0)$. Thus $I \subseteq O(I^0)$. Therefore I is an O-ideal.

$(2) \Rightarrow (3)$: Since every annihilator ideal is an α-ideal, it is clear.

$(3) \Rightarrow (4)$: Since $(x]^{**}$ is an annihilator ideal, it is obvious.

$(4) \Rightarrow (1)$: Let $x \in L$. Hence by (4), $(x]^{**} = O(F)$ for some filter F of L. Now let $t \in (x]^{**} = O(F)$. Then $t \in (y]^*$ for some $y \in F$. Hence $(x]^{**} \subseteq (y]^*$. On the other hand, we have $(y]^* \subseteq \bigcup_{y \in F} (y]^* = O(F) = (x]^{**}$. Hence we can get $(x]^{**} = (y]^*$. Therefore L is a $\star$-ADL. □

The following corollary is a direct consequence of the above theorem.

Corollary 3.6. *Let L be a $\star$-ADL. Then the following are equivalent.*

(a). Every ideal is an α-ideal
(b). Every ideal is an O-ideal
(c). Every ideal is the intersection of all minimal prime ideals containing it.

In conclusion, we would like to mention that the roles of O-ideals and annulets are most interesting in this context. Perhaps the nature of the union of annulets provides scope for further research particularly in the characterization of some more structures like normal ADLs, generalized Stone ADLs e.t.c.

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