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I-Bitopological Spaces Generated by Intuitionistic Fuzzy n-Norms

N. Thillaigovindan, Iqbal Jebril* and S. Anita Shanthi

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Abstract: In this paper we define I-bitopological space $(A, \Im_{P,Q}, \Im_{-P,-Q})$ where $\Im_{P,Q}$ and $\Im_{-P,-Q}$ are I-topologies generated by the intuitionistic fuzzy quasi pseudo n-norms P, Q and $-P, -Q$. Further a charcterization of pairwise Hausdorff I-bitopological space is also established.

Keywords: Intuitionistic fuzzy n-norms, intuitionistic fuzzy quasi pseudo n-norm, pairwise Hausdorff I-bitopological space

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1 Introduction

Motivated by the theory of n-normed linear space [8, 9, 11, 13, 15] and fuzzy normed linear space [1, 2, 3, 4, 5, 6, 7, 10, 12, 14] the notions of fuzzy n-normed linear space [16] and intuitionistic fuzzy n-normed linear space [17] have been developed. In [19,20] I-topological spaces and I-bitopological spaces generated by fuzzy norm have been discussed.

In this paper we define intuitionistic fuzzy quasi pseudo n-norm and study the I-topology and I-bitopology generated by this new norm. A characterization of I-topological spaces and I-bitopological spaces are also established.

* *Corresponding author*

2 Preliminaries

In this section we recall some useful definitions and results.

Definition 2.1. [18] A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is a *continuous t-norm* if $*$ satisfies the following conditions:
(i) $*$ is commutative and associative
(ii) $*$ is continuous
(iii) $a*1=a$, for all $a\in[0,1]$
(iv) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ and $a,b,c,d\in[0,1]=I$.

Definition 2.2. [18] A binary operation $\diamond:[0,1]\times[0,1]\to[0,1]$ is a *continuous t-co-norm* if $\diamond$satisfies the following conditions:
(i) $\diamond$ is commutative and associative
(ii) $\diamond$ is continuous
(iii) $a\diamond 0=a$, for all $a\in[0,1]$
(iv) $a\diamond b\le c\diamond d$ whenever $a\le c$ and $b\le d$ and $a,b,c,d\in[0,1]$.

Remark 2.3. [20] The algebraic operations on I can be extended pointwise to the set I^X of all maps from $X\to I$. i.e., If $\mu_1,\mu_2\in I^X$ then $(\mu_1*\mu_2)(x)=\mu_1(x)*\mu_2(x)$ for all $x\in X$.

Definition 2.4. [20] Let X be a non-empty set. A subset $\Im$ of I^X is called an *I-topology* on X if $\Im$ satisfies the following conditions:
(i) $1_X,1_\phi\in\Im$
(ii) $\mu_1,\mu_2\in\Im$ implies $\mu_1*\mu_2\in\Im$
(iii) $\{\mu_i|\ i\in$ index set$\}\subseteq\Im$ implies $\vee\mu_i\in\Im$.

Example 2.5. [20] Let $X=\{a,b\}$ and $*$ be defined by $r*s=\min\{r,s\}$. Consider $\mu_1\in I^X$ defined by $\mu_1(x)=\begin{cases}\frac{1}{2} & \text{if} \quad x=a\\ 0 & \text{if} \quad x=b\end{cases}$.
Then $\Im=\{1_X,1_\phi,\mu_1\}$ is an I-topology on X. In this example if $*$ is a product norm then $\Im=\{1_X,1_\phi,\mu_1\}$ is not an I-topology on X since $\mu_1*\mu_1$ is not an element in $\Im$.

Definition 2.6. [11] Let n $\in\mathbb{N}$ (natural numbers) and X be a real linear space of dimension greater than or equal to n. A real valued function $||\bullet,\ldots,\bullet||$ on $\underbrace{X\times\cdots\times X}_{n}=X^n$ satisfying the following four properties:

1. $||x_1, x_2, \ldots, x_n|| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent
2. $||x_1, x_2, \ldots, x_n||$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$
3. $||x_1, x_2, \ldots, kx_n|| = |k| \; ||x_1, x_2, ..., x_n||$, for any $k \in R$ (set of real numbers)
4. $||x_1, x_2, \ldots, x_{n-1}, y + z|| \leq ||x_1, x_2, \ldots, x_{n-1}, y|| + ||x_1, x_2, \ldots, x_{n-1}, z||$

is called an n-*norm* on X and the pair $(X, ||\bullet, \ldots, \bullet||)$ is called an n-*normed linear space.*

Definition 2.7. [17] An *intuitionistic fuzzy n-normed linear space* or in short i-f-n-NLS is an object of the form

$$A = \{(X, N(x_1, x_2, \ldots, x_n, t), M(x_1, x_2, \ldots, x_n, t))/(x_1, x_2, \ldots, x_n) \in X^n\}$$

where X is a linear space over a field $\mathbb{F}$, $*$ is a continuous t-norm, $\diamond$ is a continuous t-co-norm and N, M are fuzzy sets on $X^n \times (0, \infty)$; N denotes the degree of membership and M denotes the degree of non-membership of $(x_1, x_2, \ldots, x_n, t) \in X^n \times (0, \infty)$ satisfying the following conditions:

(1) $N(x_1, x_2, \ldots, x_n, t) + M(x_1, x_2, \ldots, x_n, t) \leq 1$

(2) $N(x_1, x_2, \ldots, x_n, t) > 0$

(3) $N(x_1, x_2, \ldots, x_n, t) = 1$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent

(4) $N(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$

(5) $N(x_1, x_2, \ldots, cx_n, t) = N(x_1, x_2, \ldots, x_n, \frac{t}{|c|})$ if $c \neq 0, c \in \mathbb{F}$

(6) $N(x_1, x_2, \ldots, x_n, s) * N(x_1, x_2, \ldots, x'_n, t) \leq N(x_1, x_2, \ldots, x_n + x'_n, s + t)$

(7) $N(x_1, x_2, \ldots, x_n, t) : (0, \infty) \to [0, 1]$ is continuous in t

(8) $M(x_1, x_2, \ldots, x_n, t) > 0$

(9) $M(x_1, x_2, \ldots, x_n, t) = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent

(10) $M(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$

(11) $M(x_1, x_2, \ldots, cx_n, t) = M(x_1, x_2, \ldots, x_n, \frac{t}{|c|})$, if $c \neq 0,\ c \in \mathbb{F}$

(12) $M(x_1, x_2, \ldots, x_n, s) \diamond M(x_1, x_2, \ldots, x'_n, t) \geq M(x_1, x_2, \ldots, x_n + x'_n, s + t)$

(13) $M(x_1, x_2, \ldots, x_n, t) : (0, \infty) \to [0, 1]$ is continuous in t.

Remark 2.8. For convenience we denote the intuitionistic fuzzy n-normed linear space by $A = (X, N, M, *, \diamond)$.

Example 2.9. Let $(X, ||\bullet, \ldots, \bullet||)$ be an n-normed linear space, where $X = R$. Define $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$, for all $a, b \in [0, 1]$,

$$N(x_1, x_2, \ldots, x_n, t) = e^{-||x_1, x_2, \ldots, x_n||/t},$$
$$M(x_1, x_2, \ldots, x_n, t) = 1 - e^{-||x_1, x_2, \ldots, x_n||/t}.$$

Then $A = \{(X, N(x_1, x_2, \ldots, x_n, t), M(x_1, x_2, \ldots, x_n, t))/(x_1, x_2, \ldots, x_n) \in X^n\}$ is an i-f-n-NLS.

3 I-topological and I-bitopological spaces

Definition 3.1. Let A be an i-f-n-NLS and let $\alpha \in (0, 1]$, $\epsilon > 0$ and $x \in A$. The fuzzy set $\mathbf{N}_\alpha(x, \epsilon)$ in A is defined as

$$\mathbf{N}_\alpha(x, \epsilon)(y) = \begin{cases} \alpha & \text{if } N(x - y, \epsilon) > 1 - \alpha \text{ and } M(x - y, \epsilon) < \alpha \\ 0 & \text{otherwise} \end{cases}$$

for $y \in A$ is called the *α-open sphere in an i-f-n-NLS with center at x*.

Definition 3.2. Let A be an i-f-n-NLS. A fuzzy set $\mu \in I^X$ is said to be *open* if $\mu(x) > 0$ implies there exists $\epsilon > 0$ and $\alpha \in (0, 1]$ such that $\mathbf{N}_\alpha(x, \epsilon) \subseteq \mu$.

Theorem 3.3. *Let A be an i-f-n-NLS. Then $\Im_{N,M} = \{\mu \in I^X : \mu$ is open$\}$ is an I-topology on A.*

Proof. (i) Clearly, $1_X, 1_\phi \in \Im_{N,M}$.

(ii) Proof of $\mu_1, \mu_2 \in \Im_{N,M}$ implies $\mu_1 * \mu_2 \in \Im_{N,M}$.

$\mu_1, \mu_2 \in \Im_{N,M} \Rightarrow \mu_1, \mu_2 \in I^X$ and μ_1, μ_2 are open. $\mu_1, \mu_2 \in I^X \Rightarrow \mu_1 * \mu_2 \in I^X$ (by definition of $*$). μ_1 is open. Therefore $\mu_1(x) > 0 \Rightarrow \exists\, \epsilon_1 > 0$ and $\alpha \in (0, 1]$ such that $\mathbf{N}_\alpha(x, \epsilon_1) \subseteq \mu_1$. μ_2 is open. Therefore $\mu_2(x) > 0 \Rightarrow \exists\, \epsilon_2 > 0$ and $\alpha \in (0, 1]$ such that $\mathbf{N}_\alpha(x, \epsilon_2) \subseteq \mu_2$. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$. Therefore $\mathbf{N}_\alpha(x, \epsilon) \subseteq \mu_1$ and $\mathbf{N}_\alpha(x, \epsilon) \subseteq \mu_2 \Rightarrow \mathbf{N}_\alpha(x, \epsilon) \subseteq \mu_1 * \mu_2$ (by condition (iv) in definition of $*$) $\Rightarrow \mu_1 * \mu_2$ is open. $\mu_1 * \mu_2 \in I^X$ and $\mu_1 * \mu_2$ is open $\Rightarrow \mu_1 * \mu_2 \in \Im_{N,M}$.

(iii) Let $\{\mu_i\}$ be any collection of members of $\Im_{N,M}$. Proof of $\bigcup_{i\in I} \mu_i \in \Im_{N,M}$. If $(\bigcup_{i\in I} \mu_i)(x) > 0, \exists$ an i_0, such that $\mu_{i_0}(x) > 0$. So $\exists\, \epsilon > 0$ and $\alpha \in (0,1]$ such that $\mathbf{N}_\alpha(x,\epsilon) \subseteq \mu_{i_0} \subseteq \bigcup_{i\in I} \mu_i$. Hence $\bigcup_{i\in I} \mu_i \in \Im_{N,M}$. □

Remark 3.4. $\Im_{N,M}$ is called an *I-topology on A generated by the intuitionistic fuzzy n-norms N, M* and $(A, \Im_{N,M})$ is called as an *I-topological space.*

Definition 3.5. Let $(A, \Im_{N_1,M_1})$ and $(B, \Im_{N_2,M_2})$ be two I-topological spaces. A mapping $f^{\rightarrow} : (A, \Im_{N_1,M_1}) \to (B, \Im_{N_2,M_2})$ is called *I-continuous* if $f^{\leftarrow}(v) \in \Im_{N_1,M_1}$ for all $v \in \Im_{N_2,M_2}$.

Theorem 3.6. *Let $(A, \Im_{N_1,M_1})$, $(B, \Im_{N_2,M_2})$, $(C, \Im_{N_3,M_3})$ be three I-topological spaces and $f^{\rightarrow} : (A, \Im_{N_1,M_1}) \to (B, \Im_{N_2,M_2})$, $g^{\rightarrow} : (B, \Im_{N_2,M_2}) \to (C, \Im_{N_3,M_3})$ be two I-continuous mappings. Then $g^{\rightarrow} \circ f^{\rightarrow}$ is I-continuous.*

Proof. $f^{\rightarrow} : (A, \Im_{N_1,M_1}) \to (B, \Im_{N_2,M_2})$ is I-continuous implies $f^{\leftarrow}(v) \in \Im_{N_1,M_1}$ $\forall\, v \in \Im_{N_2,M_2}$. $g^{\rightarrow} : (B, \Im_{N_2,M_2}) \to (C, \Im_{N_3,M_3})$ is I-continuous implies $g^{\leftarrow}(w) \in \Im_{N_2,M_2} \forall\, w \in \Im_{N_3,M_3}$. Now

$$\begin{aligned}(g\circ f)^{\leftarrow}(w) &= f^{\leftarrow}(g^{\leftarrow}(w))\\ &= f^{\leftarrow}(v) \in \Im_{N_1,M_1},\ \forall\, w \in \Im_{N_3,M_3}\end{aligned}$$

which implies $g^{\rightarrow} \circ f^{\rightarrow}$ is I-continuous. □

Definition 3.7. Let $\Im_{N_1,M_1}, \Im_{N_2,M_2}$ be two I-topologies on A. Then $(A, \Im_{N_1,M_1}, \Im_{N_2,M_2})$ is called an *I-bitopological space.*

Example 3.8. Let $X = \{a, b\}$. $\underbrace{X \times \cdots \times X}_{n} = \{x_1, \ldots, x_n\}, x_i$ is either a or b. We define $||x_1, x_2, \ldots, x_n|| = (\sum_{i=1}^{n} |\, x_i|^{\,2})^{\frac{1}{2}}$.

(i)
$$\begin{aligned}||x_1, x_2, \ldots, x_n|| = 0 &\Leftrightarrow (\sum_{i=1}^{n} |\, x_i|^{\,2})^{\frac{1}{2}} = 0\\ &\Leftrightarrow \sum_{i=1}^{n} |\, x_i|^{\,2} = 0\\ &\Leftrightarrow x_i = 0,\ \forall\, i = 1, 2, \ldots, n\\ &\Leftrightarrow x_1, x_2, \ldots, x_n \text{ are linearly dependent.}\end{aligned}$$

(ii) Clearly, $||x_1, x_2, \ldots, x_n||$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$.

(iii) $$\begin{aligned}||x_1, x_2, \dots, \alpha\, x_n|| &= (|\,x_1|^{\,2} + |\,x_2|^{\,2} + \cdots + |\,\alpha\, x_n|^{\,2})^{\frac{1}{2}}\\ &= (|\,x_1|^{\,2} + |\,x_2|^{\,2} + \cdots + |\,\alpha|^{\,2}|x_n|^{\,2})^{\frac{1}{2}}\\ &= |\,\alpha|\,||x_1, x_2, \dots, x_n|| \text{ if and only if } \alpha = 1.\end{aligned}$$

(iv) $$\begin{aligned}&||x_1, x_2, \dots, x_{n-1}, y|| + ||x_1, x_2, \dots, x_{n-1}, z||\\ &= (|\,x_1|^{\,2} + |\,x_2|^{\,2} + \cdots + |\,x_{n-1}|^{\,2} + |\,y|^{\,2})^{\frac{1}{2}}\\ &\quad +(|\,x_1|^{\,2} + |\,x_2|^{\,2} + \cdots + |\,x_{n-1}|^{\,2} + |\,z|^{\,2})^{\frac{1}{2}}\\ &\geq (|\,x_1|^{\,2} + |\,x_2|^{\,2} + \cdots + |\,x_{n-1}|^{\,2} + |\,y+z|^{\,2})^{\frac{1}{2}}\\ &\geq ||x_1, x_2, \dots, x_{n-1}, y+z||\end{aligned}$$

Hence $(X, ||x_1, x_2, ..., x_n||\,)$ is a n-normed linear space. Let $*, \diamond$ be defined by $r * s = \min\{r, s\}$, $r \diamond s = \max\{r, s\}$. Consider $\mu_1, \mu_2 \in I^X$ defined by

$$\mu_1(x) = \begin{cases} \frac{1}{2} & \text{if } x = a \\ 0 & \text{if } x = b \end{cases} \qquad \text{and} \qquad \mu_2(x) = \begin{cases} \frac{1}{2} & \text{if } x = a \\ \frac{1}{2} & \text{if } x = b \end{cases}.$$

Let $\Im_{N_1,M_1} = \{1_X, 1_\phi, \mu_1\}$ and $\Im_{N_2,M_2} = \{1_X, 1_\phi, \mu_2\}$. Then $(A, \Im_{N_1,M_1}, \Im_{N_2,M_2})$ is an I-bitopological space.

Definition 3.9. Let $(A, \Im_{N_1,M_1}, \Im_{N_2,M_2})$ and $(B, \Im_{N_3,M_3}, \Im_{N_4,M_4})$ be two I-bitopological spaces. Then $f^{\rightarrow} : (A, \Im_{N_1,M_1}, \Im_{N_2,M_2}) \rightarrow (B, \Im_{N_3,M_3}, \Im_{N_4,M_4})$ is I-bicontinuous if $f^{\leftarrow}(u) \in \Im_{N_1,M_1} \forall\, u \in \Im_{N_3,M_3}$ and $f^{\leftarrow}(v) \in \Im_{N_2,M_2} \forall\, v \in \Im_{N_4,M_4}$.

Definition 3.10. An I-topological space $(A, \Im_{N,M})$ is called a T_0-*space* if for every pair of distinct points $x, y \in A$, there exists $\mu \in \Im_{N,M}$ such that $\mu(x) \neq \mu(y)$.

Example 3.11. Let $X = \{a, b\}$ and $*, \diamond$ be defined by $r * s = \min\{r, s\}$, $r \diamond s = \max\{r, s\}$. Consider $\mu_1 \in I^X$ defined by $\mu_1(x) = \begin{cases} \frac{1}{2} & \text{if } x = a \\ 0 & \text{if } x = b \end{cases}$. Then $\Im_{N,M} = \{1_X, 1_\phi, \mu_1\}$ is a I-topology on A. $(A, \Im_{N,M})$ is a T_0-space, whereas $(A, \Im_{N_2,M_2})$ given in Example 3.8 is not a T_0-space.

Definition 3.12. An I-topological space $(A, \Im_{N,M})$ is called a T_1-*space* if for any two distinct points $x, y \in A$, there exists $\mu_1, \mu_2 \in \Im_{N,M}$ such that $\mu_1(x) > 0$, $\mu_1(y) = 0$ and $\mu_2(x) = 0, \mu_2(y) > 0$.

Example 3.13. Let $X = \{a, b\}$ and $*, \diamond$ be defined by $r * s = \min\{r, s\}$, $r \diamond s = \max\{r, s\}$. Consider $\mu_1, \mu_2 \in I^X$ defined by

$$\mu_1(x) = \begin{cases} \frac{1}{2} & \text{if } x = a \\ 0 & \text{if } x = b \end{cases} \qquad \text{and} \qquad \mu_2(x) = \begin{cases} 0 & \text{if } x = a \\ \frac{1}{2} & \text{if } x = b \end{cases}.$$

Then $\Im_{N,M} = \{1_X, 1_\phi, \mu_1, \mu_2\}$ is a I-topology on A and $(A, \Im_{N,M})$ is a T_1-space. The topological space given in Example 3.11 is not a T_1-space. It is clear that every T_1-space is a T_0-space but not the converse.

Definition 3.14. An I-topological space $(A, \Im_{N,M})$ is called a T_2*-space* if for any two distinct points $x, y \in A$, there exists $\mu_1, \mu_2 \in \Im_{N,M}$ such that $\mu_1(x) > 0$, $\mu_2(y) > 0$ and $\mu_1 * \mu_2 = 1_\phi$, $\mu_1 \diamond \mu_2 = 1_X$.

Example 3.15. Let $X = \{a, b\}$ and $*, \diamond$ be defined by $r * s = \max\{0, r + s - 1\}$, $r \diamond s = \min\{1, 2 - r - s\}$. Consider $\mu_1, \mu_2 \in I^X$ defined by

$$\mu_1(x) = \begin{cases} \frac{1}{2} & \text{if } x = a \\ 0 & \text{if } x = b \end{cases} \qquad \text{and} \qquad \mu_2(x) = \begin{cases} 0 & \text{if } x = a \\ \frac{1}{2} & \text{if } x = b \end{cases}.$$

Then $\Im_{N,M} = \{1_X, 1_\phi, \mu_1, \mu_2\}$ is a I-topology on A. The I-topological space $(A, \Im_{N,M})$ is a T_2-space.

Definition 3.16. An I-bitopological space $(A, \Im_{N_1,M_1}, \Im_{N_2,M_2})$ is said to be *pairwise Hausdorff* if for any two distinct points $x, y \in A$, there exists a $\Im_{N_1,M_1}$ open set μ_1 and a $\Im_{N_2,M_2}$ open set μ_2 such that $\mu_1(x) > 0, \mu_2(y) > 0$ and $\mu_1 * \mu_2 = 1_\phi$, $\mu_1 \diamond \mu_2 = 1_X$ and there exists a $\Im_{N_1,M_1}$ open set μ_3 and a $\Im_{N_2,M_2}$ open set μ_4 such that $\mu_3(y) > 0, \mu_4(x) > 0$ and $\mu_3 * \mu_4 = 1_\phi$, $\mu_3 \diamond \mu_4 = 1_X$.

Example 3.17. Let $X = \{a, b\}$ and $*, \diamond$ be defined by $r * s = \max\{0, r + s - 1\}$, $r \diamond s = \min\{1, 2 - r - s\}$. Consider $\mu_1, \mu_2, \mu_3 \in I^X$ defined by

$$\mu_1(x) = \begin{cases} \frac{1}{2} & \text{if } x = a \\ 0 & \text{if } x = b \end{cases}, \qquad \mu_2(x) = \begin{cases} 0 & \text{if } x = a \\ \frac{1}{2} & \text{if } x = b \end{cases} \qquad \text{and}$$

$$\mu_3(x) = \begin{cases} \frac{1}{2} & \text{if } x = a \\ \frac{1}{2} & \text{if } x = b \end{cases}.$$

Then $\Im_{N_1,M_1} = \{1_X, 1_\phi, \mu_1, \mu_3\}$, $\Im_{N_2,M_2} = \{1_X, 1_\phi, \mu_2, \mu_3\}$ are I-topologies on A. The I-bitopological space $(A, \Im_{N_1,M_1}, \Im_{N_2,M_2})$ is a pairwise Hausdorff space.

Definition 3.18. An I-bitopological space $(A, \Im_{N_1,M_1}, \Im_{N_2,M_2})$ is said to be *pairwise weakly Hausdorff* if for any two distinct points $x, y \in A$, there exists a $\Im_{N_1,M_1}$ open set μ_1 and a $\Im_{N_2,M_2}$ open set μ_2 such that $\mu_1(x) > 0, \mu_2(y) > 0$ and $\mu_1 * \mu_2 = 1_\phi$, $\mu_1 \diamond \mu_2 = 1_X$ or there exists a $\Im_{N_1,M_1}$ open set μ_3 and a $\Im_{N_2,M_2}$ open set μ_4 such that $\mu_3(y) > 0, \mu_4(x) > 0$ and $\mu_3 * \mu_4 = 1_\phi$, $\mu_3 \diamond \mu_4 = 1_X$.

Example 3.19. Let $X = \{a, b\}$ and $*, \diamond$ be defined by $r * s = \max\{0, r + s - 1\}$, $r \diamond s = \min\{1, 2 - r - s\}$. Consider $\mu_1, \mu_2 \in I^X$ defined by

$$\mu_1(x) = \begin{cases} \frac{1}{2} & \text{if } x = a \\ 0 & \text{if } x = b \end{cases} \qquad \text{and} \qquad \mu_2(x) = \begin{cases} \frac{1}{2} & \text{if } x = a \\ \frac{1}{2} & \text{if } x = b \end{cases}.$$

Then $\Im_{N_1,M_1} = \{1_X, 1_\phi, \mu_1\}$, $\Im_{N_2,M_2} = \{1_X, 1_\phi, \mu_2\}$ are I-topologies on A. The I-bitopological space $(A, \Im_{N_1,M_1}, \Im_{N_2,M_2})$ is a pairwise weakly Hausdorff space.

Theorem 3.20. *Assume that $\alpha \neq 0, \beta \neq 0$ implies $\alpha * \beta \neq 0$. If an I-bitopological space $(A, \Im_{N_1,M_1}, \Im_{N_2,M_2})$ is pairwise weakly Hausdorff, then $\Im_{N_1,M_1}$ and $\Im_{N_2,M_2}$ are T_0-topologies.*

Proof. Let $x, y \in A$ with $x \neq y$. Since $(A, \Im_{N_1,M_1}, \Im_{N_2,M_2})$ is pairwise weakly Hausdorff, there exists $\mu_1 \in \Im_{N_1,M_1}$ and $\mu_2 \in \Im_{N_2,M_2}$ such that $\mu_1(x) > 0$, $\mu_2(y) > 0$ and $\mu_1 * \mu_2 = 1_\phi$, $\mu_1 \diamond \mu_2 = 1_X$. Since $\mu_1(x) > 0$ and $\mu_2(y) > 0$, $\mu_1(y) = 0$ and $\mu_2(x) = 0$. Hence $\mu_1(x) > 0, \mu_1(y) = 0$ and $\mu_2(x) = 0, \mu_2(y) > 0$. That is $\Im_{N_1,M_1}$ and $\Im_{N_2,M_2}$ are T_0-topologies. □

Theorem 3.21. *Assume that $\alpha \neq 0, \beta \neq 0$ implies $\alpha * \beta \neq 0$. If an I-bitopological space $(A, \Im_{N_1,M_1}, \Im_{N_2,M_2})$ is pairwise Hausdorff, then $\Im_{N_1,M_1}$ and $\Im_{N_2,M_2}$ are T_1-topologies.*

Proof. Let $x, y \in A$ with $x \neq y$. Since $(A, \Im_{N_1,M_1}, \Im_{N_2,M_2})$ is pairwise Hausdorff, $\exists\, \mu_1 \in \Im_{N_1,M_1}$ and $\mu_2 \in \Im_{N_2,M_2}$ such that $\mu_1(x) > 0, \mu_2(y) > 0$ and $\mu_1 * \mu_2 = 1_\phi$, $\mu_1 \diamond \mu_2 = 1_X$. Also there exists $\mu_3 \in \Im_{N_1,M_1}$ and $\mu_4 \in \Im_{N_2,M_2}$ such that $\mu_3(y) > 0, \mu_4(x) > 0$ and $\mu_3 * \mu_4 = 1_\phi$, $\mu_3 \diamond \mu_4 = 1_X$. Hence $\mu_1, \mu_3 \in \Im_{N_1,M_1}$ with $\mu_1(x) > 0, \mu_1(y) = 0$ and $\mu_3(x) = 0, \mu_3(y) > 0$. Also $\mu_2, \mu_4 \in \Im_{N_2,M_2}$ with $\mu_2(x) = 0, \mu_2(y) > 0$ and $\mu_4(x) > 0, \mu_4(y) = 0$. Therefore $\Im_{N_1,M_1}$ and $\Im_{N_2,M_2}$ are T_1-topologies. □

Theorem 3.22. *Assume that $\alpha \neq 0, \beta \neq 0$ implies $\alpha * \beta \neq 0$. If an I-bitopological space $(A, \Im_{N_1,M_1}, \Im_{N_2,M_2})$ is pairwise Hausdorff, then $\Im_{N_1,M_1}$ or $\Im_{N_2,M_2}$ is a T_2-topology.*

Proof. Let $x, y \in A$ with $x \neq y$. Since $(A, \Im_{N_1,M_1}, \Im_{N_2,M_2})$ is pairwise Hausdorff, $\exists\ \mu_1 \in \Im_{N_1,M_1}$ and $\mu_2 \in \Im_{N_2,M_2}$ such that $\mu_1(x) > 0, \mu_2(y) > 0$ and $\mu_1 * \mu_2 = 1_\phi$, $\mu_1 \diamond \mu_2 = 1_X$. Also there exists $\mu_3 \in \Im_{N_1,M_1}$ and $\mu_4 \in \Im_{N_2,M_2}$ such that $\mu_3(y) > 0$, $\mu_4(x) > 0$ and $\mu_3 * \mu_4 = 1_\phi$, $\mu_3 \diamond \mu_4 = 1_X$. Since $\mu_1(x) > 0$ and $\mu_2(y) > 0$, $\mu_1(y) = 0$ and $\mu_2(x) = 0$. Similarly, $\mu_4(y) = 0, \mu_3(x) = 0$. Therefore we have $(\mu_1 * \mu_3)(x) = 0$, $(\mu_1 * \mu_3)(y) = 0$ and $(\mu_2 * \mu_4)(x) = 0$, $(\mu_2 * \mu_4)(y) = 0$. Also $(\mu_1 \diamond \mu_3)(x) = 1, (\mu_1 \diamond \mu_3)(y) = 1$ and $(\mu_2 \diamond \mu_4)(x) = 1$, $(\mu_2 \diamond \mu_4)(y) = 1$. Suppose there is a $z \neq x, y$ and $(\mu_1 * \mu_3)(z) \neq 0$. Then $\mu_1(z) \neq 0, \mu_3(z) \neq 0$. Hence $\mu_2(z) = 0$ and $\mu_4(z) = 0$ and so we conclude that there exists $\mu_5, \mu_6 \in \Im_{N_1,M_1}$ with $\mu_5(x) > 0, \mu_5(y) = 0$ and $\mu_6(y) > 0, \mu_6(x) = 0$. Therefore $(\mu_5 * \mu_6)(x) = 0$, $(\mu_5 \diamond \mu_6)(y) = 0$, so that $\mu_5 * \mu_6 = 1_\phi$. Hence $\mu_5 \diamond \mu_6 = 1_X$. Therefore $\Im_{N_1,M_1}$ is a T_2-topology. □

Theorem 3.23. *Assume that $\alpha \neq 0, \beta \neq 0$ implies $\alpha * \beta \neq 0$. If either $\Im_{N_1,M_1}$ or $\Im_{N_2,M_2}$ is a T_2-topology on A and the other is a T_1-topology on A, then $(A, \Im_{N_1,M_1}, \Im_{N_2,M_2})$ is a pairwise weakly Hausdorff space.*

Proof. Suppose $\Im_{N_1,M_1}$ is a T_2-topology on A and $\Im_{N_2,M_2}$ is a T_1-topology on A. Let $x, y \in A$ with $x \neq y$. Then there exists $\mu_1, \mu_2 \in \Im_{N_1,M_1}$ such that $\mu_1(x) > 0, \mu_2(y) > 0$ and $\mu_1 * \mu_2 = 1_\phi$, $\mu_1 \diamond \mu_2 = 1_X$. Also there exists $\mu_3, \mu_4 \in \Im_{N_2,M_2}$ such that $\mu_3(x) > 0, \mu_3(y) = 0$ and $\mu_4(x) = 0, \mu_4(y) > 0$. Hence $\mu_1(x) > 0, \mu_4(y) > 0$ and $\mu_3(x) > 0, \mu_2(y) > 0$. Therefore we have $(\mu_1 * \mu_4)(x) = 0, (\mu_1 * \mu_4)(y) = 0$ and $(\mu_3 * \mu_2)(x) = 0, (\mu_3 * \mu_2)(y) = 0$. Also $(\mu_1 \diamond \mu_4)(x) = 1, (\mu_1 \diamond \mu_4)(y) = 1$ and $(\mu_3 \diamond \mu_2)(x) = 1, (\mu_3 \diamond \mu_2)(y) = 1$. Let $z \neq x, y$ with $(\mu_1 * \mu_4)(z) \neq 0$. Then $\mu_1(z) \neq 0, \mu_4(z) \neq 0$. Hence $\mu_2(z) = 0$ and so $(\mu_3 * \mu_2)(z) = 0$. Therefore we can find $\mu_5 \in \Im_{N_1,M_1}$ and $\mu_6 \in \Im_{N_2,M_2}$ with $\mu_5(x) > 0, \mu_6(y) > 0$ such that $\mu_5 * \mu_6 = 1_\phi, \mu_5 \diamond \mu_6 = 1_X$ as proved earlier or $\mu_1 * \mu_4 = 1_\phi, \mu_1 \diamond \mu_4 = 1_X$ and $(A, \Im_{N_1,M_1}, \Im_{N_2,M_2})$ is pairwise weakly Hausdorff. □

4 Intuitionistic fuzzy quasi pseudo n-normed linear spaces

Definition 4.1. Let X be any vector space, $*$ be a continuous t-norm and $\diamond$ a continuous t-co-norm. Then the functions $P, Q : X^n \times (0, \infty) \to [0, 1]$ satisfying the following conditions

(1) $P(0,t) + Q(0,t) = 1$ where $0 = (0,0,\dots,0)$

(2) $P(x_1, x_2, \dots, x_n - x'_n, t+s) \geq P(x_1, x_2, \dots, x_n, t) * P(x_1, x_2, \dots, x'_n, s)$

(3) $P(x_1, x_2, \dots, x_n, \cdot) : (0,\infty) \to [0,1]$ is left continuous

(4) $P(x_1, x_2, \dots, x_n, t) \to 1$ as $t \to \infty$

(5) $Q(x_1, x_2, \dots, x_n - x'_n, t+s) \leq Q(x_1, x_2, \dots, x_n, t) \diamond Q(x_1, x_2, \dots, x'_n, s)$

(6) $Q(x_1, x_2, \dots, x_n, \cdot) : (0,\infty) \to [0,1]$ is left continuous

(7) $Q(x_1, x_2, \dots, x_n, t) \to 0$ as $t \to \infty$

for all $x_1, x_2, \dots, x_n, x'_n \in X, t, s \in (0,\infty)$ is called an *intuitionistic fuzzy quasi pseudo n-norm on X* and $(X, P, Q, *, \diamond)$ is called an *intuitionistic fuzzy quasi pseudo n-normed linear space* or in short i-f-q-p-n-NLS.

Example 4.2. Let X be any real vector space, $a*b = \min\{a,b\}, a\diamond b = \max\{a,b\}$. Define

$$P(x_1, x_2, ..., x_n, t) = \begin{cases} 0 & \text{if } (x_1, x_2, \dots, x_n) \neq 0 \text{ and } t \in (0,1] \\ 1 - \frac{1}{t} & \text{if } (x_1, x_2, \dots, x_n) \neq 0 \text{ and } t \in (1,\infty) \\ 1 & \text{if } (x_1, x_2, \dots, x_n) = 0 \text{ and } t \in (0,\infty) \end{cases}$$

and

$$Q(x_1, x_2, ..., x_n, t) = \begin{cases} 1 & \text{if } (x_1, x_2, \dots, x_n) \neq 0 \text{ and } t \in (0,1] \\ \frac{1}{t} & \text{if } (x_1, x_2, \dots, x_n) \neq 0 \text{ and } t \in (1,\infty) \\ 0 & \text{if } (x_1, x_2, \dots, x_n) = 0 \text{ and } t \in (0,\infty) \end{cases}.$$

(i) Clearly $P(0,t) + Q(0,t) = 1$.

(ii) Since $\frac{1}{t+s} < \frac{1}{t}$ and $\frac{1}{t+s} < \frac{1}{s}, 1 - \frac{1}{t+s} \geq 1 - \frac{1}{t} * 1 - \frac{1}{s}$ for all $t, s > 0$.
Hence $P(x_1, x_2, \dots, x_n - x'_n, t+s) \geq P(x_1, x_2, \dots, x_n, t) * Q(x_1, x_2, \dots, x'_n, s)$.

(iii) $P(x_1, x_2, \dots, x_n, \cdot) : (0,\infty) \to [0,1]$ is left continuous.

(iv) $P(x_1, x_2, \dots, x_n, t) \to 1$ as $t \to \infty$.

(v) Since $\frac{1}{t+s} \leq \frac{1}{t} \diamond \frac{1}{s}$,

$$Q(x_1, x_2, \dots, x_n - x'_n, t+s) \leq Q(x_1, x_2, \dots, x_n, t) \diamond Q(x_1, x_2, \dots, x'_n, s).$$

(vi) $Q(x_1, x_2, \ldots, x_n, \cdot) : (0, \infty) \to [0, 1]$ is left continuous.

(vii) $Q(x_1, x_2, \ldots, x_n, t) \to 0$ as $t \to \infty$.

Hence $(X, P, Q, *, \diamond)$ is an i-f-q-p-n-NLS. Also $P((x_1/5, x_2, \ldots, x_n), 4/5) = 0$ and $P((x_1, x_2, \ldots, x_n), (4/5)/|1/5|) = 3/4$. Therefore $P(kx_1, x_2, \ldots, x_n, t) \neq P(x_1, x_2, \ldots, x_n, t/|k|)$ for $t = 4/5$ and $k = 1/5$. Hence $(X, P, Q, *, \diamond)$ is not an i-f-n-NLS.

Definition 4.3. An i-f-q-p-n-norm P, Q is said to be an *intuitionistic fuzzy quasi n-norm* if $P(x_1, x_2, \ldots, x_n, t) = 1$ and $Q(x_1, x_2, \ldots, x_n, t) = 0$, $\forall\ t$ implies $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$.

Definition 4.4. An i-f-q-p-n-norm P, Q is said to be an *intuitionistic fuzzy pseudo n-norm* if $P(x_1, x_2, \ldots, kx_n, t) = P(x_1, x_2, \ldots, x_n, \frac{t}{|k|})$ and $Q(x_1, x_2, \ldots, kx_n, t) = Q(x_1, x_2, \ldots, x_n, \frac{t}{|k|})$ for all scalar k and $(x_1, x_2, \ldots, x_n) \in X^n$.

Remark 4.5. $P(0, 0, \ldots, k0, t) = P(0, 0, \ldots, 0, \frac{t}{|k|}) = 1$ and $Q(0, 0, \ldots, k0, t) = Q(0, 0, \ldots, 0, \frac{t}{|k|}) = 0$, i.e., $P(0, s) = 1$ and $Q(0, s) = 0$ where s is positive.

Proposition 4.6. *Let P, Q be i-f-q-p-n-norm on X and suppose*

$$P_1(x_1, x_2, \ldots, x_n, t) = P(x_1, x_2, \ldots, -x_n, t),$$
$$Q_1(x_1, x_2, \ldots, x_n, t) = Q(x_1, x_2, \ldots, -x_n, t)$$

where $(x_1, x_2, \ldots, x_n) \in X^n$. Then P_1, Q_1 is also an i-f-q-p-n-norm on X.

Proof. (i) $P_1(0, t) = P(0, t) = 1$ and $Q_1(0, t) = Q(0, t) = 0$ where $0 = (0, 0, \ldots, 0)$.

(ii) $P_1(x_1, x_2, \ldots, x_n - x'_n, t + s)$

$$\begin{aligned} &= P(x_1, x_2, \ldots, x'_n - x_n, t + s) \\ &= P(x_1, x_2, \ldots, -x_n - (-x'_n), t + s) \\ &\geq P(x_1, x_2, \ldots, -x_n, t) * P(x_1, x_2, \ldots, -x'_n, s) \\ &\geq P_1(x_1, x_2, \ldots, x_n, t) * P_1(x_1, x_2, \ldots, x'_n, s). \end{aligned}$$

Similarly, $Q_1(x_1, x_2, \ldots, x_n - x'_n, t + s)$
$\leq Q(x_1, x_2, \ldots, x_n, t) \diamond Q(x_1, x_2, \ldots, x'_n, s)$.

(iii) Since $P(x_1, x_2, \ldots, x_n, \cdot), Q(x_1, x_2, \ldots, x_n, \cdot) : (0, \infty) \to [0, 1]$ is left continuous, $P_1(x_1, x_2, \ldots, x_n, \cdot)$ and $Q_1(x_1, x_2, \ldots, x_n, \cdot) : (0, \infty) \to [0, 1]$ is also left continuous.

(iv) Also $P_1(x_1, x_2, \ldots, x_n, t) \to 1$ and $Q_1(x_1, x_2, \ldots, x_n, t) \to 0$ as $t \to \infty$. Therefore P_1, Q_1 is also an i-f-q-p-n-norm on X.

□

Remark 4.7. P_1, Q_1 defined by $P_1(x_1, x_2, \ldots, x_n, t) = P(x_1, x_2, \ldots, -x_n, t)$, $Q_1(x_1, x_2, \ldots, x_n, t) = Q(x_1, x_2, \ldots, -x_n, t)$ are called *conjugate i-f-q-p-n-norm* of P, Q. If P, Q is an intuitionistic fuzzy pseudo n-norm, then $P = P_1$ and $Q = Q_1$. Again if P, Q is an intuitionistic fuzzy quasi n-norm, then so is P_1, Q_1. Hereafter we denote the conjugate i-f-q-p-n-norm of P, Q by $-P, -Q$.

Definition 4.8. A function $' : [0, 1] \to [0, 1]$ is said to be an *order reverting involution on* $[0, 1]$ if it satisfies the following conditions

(i) $\alpha \leq \beta \Rightarrow \beta' \leq \alpha'$

(ii) $\alpha'' = \alpha$ for $\alpha, \beta \in [0, 1]$.

Definition 4.9. Let A be an i-f-n-NLS along with an order reverting involution $'$ on I and $\alpha \in (0, 1], \epsilon > 0$ and $x \in A$. The fuzzy set $\mathbf{N}'_\alpha(x, \epsilon) \in I^X$ is defined as

$$\mathbf{N}'_\alpha(x, \epsilon)(y) = \begin{cases} \alpha & \text{if } N(x - y, \epsilon) > \alpha' \text{ and } M(x - y, \epsilon) < 1 - \alpha' \\ 0 & \text{otherwise} \end{cases}$$

where $y \in A$ is called the *α-open sphere in an i-f-n-NLS with an order reverting involution $'$ on I and centre x.*

Definition 4.10. Let A be an i-f-n-NLS with an order reverting involution $'$ on I. A fuzzy set $\mu \in I^X$ is said to be *open* if $\mu(x) > 0$ implies there exists $\epsilon > 0$ and $\alpha \in (0, 1]$ such that $\mathbf{N}'_\alpha(x, \epsilon) \subseteq \mu$.

Note 4.11. For the rest of the paper we consider only t-norms for which $\alpha \neq 0$, $\beta \neq 0$ implies $\alpha * \beta \neq 0$.

Theorem 4.12. *Let A be an i-f-n-NLS with an order reverting involution $'$ on I. Then $\Im_{N', M'} = \{\mu \in I^X : \mu \text{ is open}\}$ is an I-topology on A.*

Proof. Proof of this Theorem is obvious. □

Theorem 4.12 implies that an i-f-n-norm generates an I-topology. The i-f-q-p-n-norm is a weak form of an i-f-n-norm, but still it generates an I-topology as shown by the following.

Theorem 4.13. *Let $(X, P, Q, *, \diamond)$ be an i-f-q-p-n-NLS along with an order reverting involution $'$ on I. Then the collection $\Im_{P,Q} = \{\mu \in I^X : \mu \text{ is open}\}$ is an I-topology on $(X, P, Q, *, \diamond)$.*

Proof. (i) Clearly $1_X, 1_\phi \in \Im_{P,Q}$.

(ii) Let $\mu_1, \mu_2 \in \Im_{P,Q}$ and suppose there exists an element x such that $(\mu_1 * \mu_2)(x) > 0$. Then $\mu_1(x) > 0$ and $\mu_2(x) > 0$, where $x = (x_1, x_2, \ldots, x_n) \in X^n$, i.e., there are $\alpha_1, \alpha_2 \in (0,1]$ and $\epsilon_1, \epsilon_2 > 0 \ni \mathbf{N}'_{\alpha_1}(x, \epsilon_1) \subseteq \mu_1$ and $\mathbf{N}'_{\alpha_2}(x, \epsilon_2) \subseteq \mu_2$. Consider $\alpha = \alpha_1 * \alpha_2$ and $\epsilon = \min\{\epsilon_1, \epsilon_2\}$.

Since $\alpha' \geq \alpha'_1, P(x_1, x_2, \ldots, x_n, \epsilon) > \alpha'$ and $Q(x_1, x_2, \ldots, x_n, \epsilon) < 1 - \alpha'$
$\Rightarrow P(x_1, x_2, \ldots, x_n, \epsilon) > \alpha'_1$ and $Q(x_1, x_2, \ldots, x_n, \epsilon) < 1 - \alpha'_1$.
Since $\epsilon \leq \epsilon_1$, $P(x_1, x_2, \ldots, x_n, \epsilon_1) > \alpha'_1$ and $Q(x_1, x_2, \ldots, x_n, \epsilon_1) < 1 - \alpha'_1$ and hence $\mathbf{N}'_\alpha(x, \epsilon) \subseteq \mathbf{N}'_{\alpha_1}(x, \epsilon_1) \subseteq \mu_1$ and $\mathbf{N}'_\alpha(x, \epsilon) \subseteq \mathbf{N}'_{\alpha_2}(x, \epsilon_2) \subseteq \mu_2$, which implies that $\mathbf{N}'_\beta(x, \epsilon) \subseteq \mathbf{N}'_\alpha(x, \epsilon) * \mathbf{N}'_\alpha(x, \epsilon) \subseteq \mu_1 * \mu_2$ where $\beta = \alpha * \alpha$. Hence $\mu_1, \mu_2 \in \Im_{P,Q}$ implies $\mu_1 * \mu_2 \in \Im_{P,Q}$.

(iii) Let $\{\mu_i\}$ be any collection of members of $\Im_{P,Q}$. If $\vee\mu_i(x) > 0$, then $\mu_j(x) > 0$ for some j. Hence $\exists\, \alpha \in (0,1]$ and $\epsilon > 0 \ni \mathbf{N}'_\alpha(x, \epsilon) \subseteq \mu_j \subseteq \vee\mu_i$. Thus $\Im_{P,Q}$ is an I-topology on $(X, P, Q, *, \diamond)$.

□

Proposition 4.14. *Let $(X, P, Q, *, \diamond)$ be an i-f-q-p-n-NLS along with an order reverting involution $'$ on I. The fuzzy set μ in I^X is open if and only if μ is the union of open sets in I^X.*

Proof. Let $\mu \in \Im_{P,Q}$ and $\mu(x) > 0$. Then there exists $\alpha \in (0,1], \epsilon > 0$ and $x \in A \ni \mathbf{N}'_\alpha(x, \epsilon) \subseteq \mu$. Consider $\mathbf{N}^\circ_\alpha(x, \epsilon)$ defined by

$$\mathbf{N}^\circ_\alpha(x, \epsilon)(y) = \begin{cases} \mu(x) & \text{if } P(x - y, \epsilon) > \alpha' \text{ and } Q(x - y, \epsilon) < 1 - \alpha' \\ 0 & \text{otherwise} \end{cases}$$

where $y \in A$. Then clearly $\mu = \vee\mathbf{N}^\circ_\alpha(x, \epsilon)$ and each $\mathbf{N}^\circ_\alpha(x, \epsilon)$ is an open set. □

Theorem 4.15. *Let $(X, P, Q, *, \diamond)$ be an i-f-q-p-n-NLS along with an order reverting involution $'$ on I. Then P, Q is an intuitionistic fuzzy quasi n-norm on A if and only if $(A, \Im_{P,Q})$ is a T_1-space.*

Proof. If P, Q is an intuitionistic fuzzy quasi n-norm on A, then for all $x, y \in A$ with $x \neq y$, we have $P(x-y, t) = r$ and $Q(x-y, t) = 1-r$ for some $0 < r < 1$ and for some $t > 0$. Now it is possible to choose one s such that $s' > r$. Consider the s-open spheres $\mu_1 = \mathbf{N}'_s(x, \frac{t}{2})$ and $\mu_2 = \mathbf{N}'_s(y, \frac{t}{2})$. We claim that $\mathbf{N}'_s(x, \frac{t}{2})(y) = 0$ and $\mathbf{N}'_s(y, \frac{t}{2})(x) = 0$. If not $P(x-y, \frac{t}{2}) > s'$ and $Q(x-y, \frac{t}{2}) < 1-s'$, i.e.,

$$\begin{aligned} P(x-y,t) &= P((x-y)+0,t) \\ &\geq P(x-y,\frac{t}{2}) * P(0,\frac{t}{2}) \\ &> s' > r \text{ and} \\ Q(x-y,t) &\leq Q(x-y,\frac{t}{2}) \diamond Q(0,\frac{t}{2}) \\ &< 1-s' < 1-r \text{ which is a contradiction.} \end{aligned}$$

Also $\mu_1(x) = s$ and $\mu_2(y) = s > 0$. Hence $(A, \Im_{P,Q})$ is a T_1-space.

Conversely, suppose $(A, \Im_{P,Q})$ is a T_1-space. Take $x, y \in A$ with $x \neq y$. Then there exists $\mu_1, \mu_2 \in \Im_{P,Q}$ such that $\mu_1(x) > 0, \mu_1(y) = 0$ and $\mu_2(y) > 0$, $\mu_2(x) = 0$. Hence $\exists\, s_1, s_2 \in (0,1]$ and $t_1, t_2 > 0 \ni \mathbf{N}'_{s_1}(x, t_1) \subseteq \mu_1$ and $\mathbf{N}'_{s_2}(y, t_2) \subseteq \mu_2$. Now since $\mu_1(y) = 0, P(x-y, t_1) \leq s_1$ and $Q(x-y, t_1) \geq 1-s_1$. Similarly $P(y-x, t_2) \leq s_2$ and $Q(y-x, t_2) \geq 1-s_2$. Hence $P(x-y, t) \neq 1$ and $Q(x-y, t) \neq 0$. This means that, suppose $x \neq 0$, then there is a $t > 0$ such that $P(x-0, t) \neq 1$ and $Q(x-0, t) \neq 0$ where $0 = (0, 0, \ldots, 0)$. Hence $P(x, t) = 1$ and $Q(x, t) = 0$ if and only if $x = 0$. $\square$

Theorem 4.16. *Let $(A, \Im_{P,Q}, \Im_{-P,-Q})$ be an I-bitopological space generated by the conjugate pairs of i-f-q-p-n-norms P, Q and $-P, -Q$ along with an order reverting involution $'$ on I. If P, Q is an intuitionistic fuzzy quasi n-norm, then $(A, \Im_{P,Q}, \Im_{-P,-Q})$ is a pairwise Hausdorff space.*

Proof. Since P, Q is an intuitionistic fuzzy quasi n-norm, $-P, -Q$ is also an intuitionistic fuzzy quasi n-norm. Hence $\Im_{P,Q}, \Im_{-P,-Q}$ are T_1-spaces. Let $x, y \in A$ with $x \neq y$. Since $\Im_{P,Q}$ is a T_1-space, there exists $\mu_1, \mu_2 \in \Im_{P,Q} \ni \mu_1(x) > 0$, $\mu_1(y) = 0$ and $\mu_2(y) > 0, \mu_2(x) = 0$. Similarly $\exists\, \mu_3, \mu_4 \in \Im_{-P,-Q}$ such that $\mu_3(x) > 0, \mu_3(y) = 0$ and $\mu_4(x) = 0, \mu_4(y) > 0$. Now $\exists\, \alpha_1, \alpha_2 \in (0,1]$ and

$\epsilon_1, \epsilon_2 > 0 \ni \mathbf{N}'_{\alpha_1}(x, \epsilon_1) \subseteq \mu_1$ and $\mathbf{N}'_{\alpha_2}(x, \epsilon_2) \subseteq \mu_3$. Since $\mathbf{N}'_{\alpha_1}(x, \epsilon_1)(y) = 0, P(x-y, \epsilon_1) \leq \alpha'_1$ and $Q(x-y, \epsilon_1) \geq 1-\alpha'_1$.

Similarly $-P(x-y, \epsilon_2) \leq \alpha'_2$ and $-Q(x-y, \epsilon_2) \geq 1-\alpha'_2$. Let $\alpha = \alpha_1 * \alpha_2$ and $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Now it is possible to choose one $0 < s < 1$ such that $s' * s' > \alpha'$. Consider the s-open spheres $\mathbf{N}'_s(x, \frac{\epsilon}{2})$ in $\Im_{P,Q}$ and $\mathbf{N}'_s(y, \frac{\epsilon}{2})$ in $\Im_{-P,-Q}$. Then it is enough to prove $\mathbf{N}'_s(x, \frac{\epsilon}{2}) * \mathbf{N}'_s(y, \frac{\epsilon}{2}) = 1_\phi$.

Suppose $\mathbf{N}'_s(x, \frac{\epsilon}{2}) * \mathbf{N}'_s(y, \frac{\epsilon}{2})(z) > 0$ for some $z \in A$, then $\mathbf{N}'_s(x, \frac{\epsilon}{2})(z) > 0$ and $\mathbf{N}'_s(y, \frac{\epsilon}{2})(z) > 0$. Hence $P(x-z, \frac{\epsilon}{2}) > s'$, $Q(x-z, \frac{\epsilon}{2}) < 1-s'$ and $-P(y-z, \frac{\epsilon}{2}) > s', -Q(y-z, \frac{\epsilon}{2}) < 1-s'$. Also it is possible to choose δ such that $0 < \delta < \frac{\epsilon}{2}$ and $P(x-z, \delta) > s', Q(x-z, \delta) < 1-s'$. Now

$$\begin{aligned}
-P(x-y, \frac{\epsilon}{2}) &\geq -P(x-y, \epsilon) \\
&\geq -P((x-z)-(y-z), \epsilon) \\
&\geq -P(x-z, \frac{\epsilon}{2}) * -P(y-z, \frac{\epsilon}{2}) \\
&\geq P(z-x, \frac{\epsilon}{2}) * -P(y-z, \frac{\epsilon}{2}) \\
&\geq P(0-(x-z), \delta + (\frac{\epsilon}{2} - \delta)) * -P(y-z, \frac{\epsilon}{2}) \\
&\geq P(0, \frac{\epsilon}{2} - \delta) * P(x-z, \delta) * (-P(y-z, \frac{\epsilon}{2})) \\
&= 1 * P(x-z, \delta) * (-P(y-z, \frac{\epsilon}{2})) \\
&= P(x-z, \delta) * (-P(y-z, \frac{\epsilon}{2})) \\
&> s' * s' > \alpha'.
\end{aligned}$$

Hence $-P(x-y, \frac{\epsilon}{2}) > \alpha'$. Therefore $-P(x-y, \frac{\epsilon}{2}) > \alpha'_2$. Similarly $-Q(x-y, \frac{\epsilon}{2}) < 1-\alpha'_2$, which is a contradiction.

Similarly there is a s-open sphere $\mathbf{N}'_s(y, \frac{\epsilon}{2})$ in $\Im_{P,Q}$ and $\mathbf{N}'_s(x, \frac{\epsilon}{2})$ in $\Im_{-P,-Q}$ such that $\mathbf{N}'_s(y, \frac{\epsilon}{2}) * \mathbf{N}'_s(x, \frac{\epsilon}{2}) = 0$. □

Theorem 4.17. *Let $(X, P, Q, *, \diamond)$ be an i-f-q-p-n-NLS along with an order inverting involution $'$ on I. Then $(A, \Im_{P,Q})$ is a T_2-space if and only if P, Q is an intuitionistic fuzzy quasi n-norm on A.*

Proof. Suppose P, Q is an intuitionistic fuzzy quasi n-norm on A. If $x, y \in A$ with $x \neq y$, then $P(x-y, t) \neq 1$ and $Q(x-y, t) \neq 0$ for some t. Suppose $P(x-y, t) = r$ and $Q(x-y, t) = 1-r$ where $0 < r < 1$. Now choose $s > 0$ such that $s' * s' > r$. Then $\mathbf{N}'_s(x, \frac{t}{2}) * \mathbf{N}'_s(y, \frac{t}{2}) = 1_\phi$. Hence $(A, \Im_{P,Q})$ is a T_2-space.

Conversely, suppose that $(A, \Im_{P,Q})$ is a T_2-space. Let $x \neq y$ in A. Then there exists $\Im_{P,Q}$ open sets μ_1, μ_2 such that $\mu_1(x) > 0, \mu_2(y) > 0$ with $\mu_1 * \mu_2 = 1_\phi, \mu_1 \diamond \mu_2 = 1_X$. Since $\mu_1(x) > 0, \mu_2(x) = 0$ and $\mu_2(y) > 0, \mu_1(y) = 0$. Hence $(A, \Im_{P,Q})$ is a T_1-space and so by Theorem 4.15 P, Q is an intuitionistic fuzzy quasi n-norm on A. □

Note 4.18. In general a T_2-space need not be a T_1-space. However if P, Q is an intuitionistic fuzzy quasi n-norm, then $(A, \Im_{P,Q})$ is a T_2-space as well as a T_1-space.

Theorem 4.19. *Let $(X, P, Q, *, \diamond)$ be an i-f-q-p-n-NLS along with an order inverting involution $'$ on I. The i-f-q-p-n-norm P, Q is an intuitionistic fuzzy quasi n-norm if and only if the I-bitopological space $(A, \Im_{P,Q}, \Im_{-P,-Q})$ is pairwise Hausdorff.*

Proof. Suppose $(A, \Im_{P,Q}, \Im_{-P,-Q})$ is a pairwise Hausdorff space. Then $\Im_{P,Q}$ and $\Im_{-P,-Q}$ are T_1-topologies. Hence P, Q and $-P, -Q$ are intuitionistic fuzzy quasi n-norms on A.

Conversely, suppose P, Q is an intuitionistic fuzzy quasi n-norm, then by Theorem 4.16, the I-bitopological space $(A, \Im_{P,Q}, \Im_{-P,-Q})$ is pairwise Hausdorff. □

Theorem 4.20. *If P, Q and $-P, -Q$ are two i-f-q-p-n-norms on A, then the I-bitopological space $(A, \Im_{P,Q}, \Im_{-P,-Q})$ is pairwise weakly Hausdorff if and only if P, Q and $-P, -Q$ are intuitionistic fuzzy quasi n-norms.*

Proof. Suppose $(A, \Im_{P,Q}, \Im_{-P,-Q})$ is pairwise weakly Hausdorff space, then $\Im_{P,Q}$ and $\Im_{-P,-Q}$ are T_1-topologies. Hence P, Q and $-P, -Q$ are intuitionistic fuzzy quasi n-norms on A.

Conversely, suppose P, Q and $-P, -Q$ are intuitionistic fuzzy quasi n-norms on A, then $\Im_{P,Q}$ and $\Im_{-P,-Q}$ are T_2-topologies. Let x and y be two distinct points in A. Then $\exists\ \mu_1(x) > 0$ and $\mu_2(y) > 0$ such that $\mu_1 * \mu_2 = 1_\phi$ and $\exists\ \mu_3$ and μ_4 in $\Im_{P_1,Q_1}$ with $\mu_3(x) > 0$ and $\mu_4(y) > 0$ such that $\mu_3 * \mu_4 = 1_\phi$. It is enough if we show that $\mu_1 * \mu_4 = 1_\phi$ or $\mu_2 * \mu_3 = 1_\phi$. Clearly $(\mu_1 * \mu_4)(x) = 0$, $(\mu_1 * \mu_4)(y) = 0, (\mu_2 * \mu_3)(x) = 0, (\mu_2 * \mu_3)(y) = 0$. Suppose there exists an element $z \in A$ such that $(\mu_1 * \mu_4)(z) \neq 0$. Then $\mu_1(z) > 0$ and $\mu_4(z) > 0$ and so $\mu_2(z) = 0, \mu_3(z) = 0$ and hence we conclude that $\mu_1 * \mu_4 = 1_\phi$ or $\mu_2 * \mu_3 = 1_\phi$. □

Note 4.21. In general, a pairwise weakly Hausdorff space need not be a pairwise Hausdorff space. However if P, Q is an intuitionistic fuzzy quasi n-norm, then

$(A, \Im_{P,Q}, \Im_{-P,-Q})$ is a pairwise weakly Hausdorff as well as a pairwise Hausdorff space as proved in the following.

Theorem 4.22. *$(A, \Im_{P,Q}, \Im_{-P,-Q})$ is a pairwise weakly Hausdorff space if and only if $(A, \Im_{P,Q}, \Im_{-P,-Q})$ is a pairwise Hausdorff space.*

Proof. Suppose $(A, \Im_{P,Q}, \Im_{-P,-Q})$ is a pairwise weakly Hausdorff space, then $\Im_{P,Q}$ or $\Im_{-P,-Q}$ is a T_2-space. Hence by Theorem 4.17, P, Q is an intuitionistic fuzzy quasi n-norm and so by Theorem 4.19 $(A, \Im_{P,Q}, \Im_{-P,-Q})$ is a pairwise Hausdorff space.

Conversely, suppose $(A, \Im_{P,Q}, \Im_{-P,-Q})$ is a pairwise Hausdorff space, then trivially it is a pairwise weakly Hausdorff space. □

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N. Thillaigovindan
Department of Mathematics,

Annamalai University,
Annamalainagar-608002, Tamilnadu, India
Email: thillai_n@sify.com

Iqbal Jebril
Department of Mathematics,
Faculty of Science,
King Faisal Universty,
P.O.Box 2600- Hufuf, Hasa 31982, Saudi Arabia
Email: iqbal501 hotmail.com

S. Anita Shanthi
Department of Mathematics,
Annamalai University,
Annamalainagar-608002, Tamilnadu, India
Email: shanthi.anita@yahoo.com