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A Refined Criterion for Schanuel's Conjecture

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Abstract: We extend the range of validity for the parameters in the equivalence between Roy's Conjecture and Schanuel's Conjecture.

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1 Conjectures and Main Result

One of the main open problems in transcendental number theory is the Schanuel's Conjecture ([1] Chapter 3: Historical note) which was stated in the 1960's.

Conjecture 1.1 (Schanuel). *Let ℓ be a positive integer and let $y_1, \ldots, y_\ell \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then*

$$\mathrm{tr.deg}_{\mathbb{Q}} \mathbb{Q}(y_1, \ldots, y_\ell, e^{y_1}, \ldots, e^{y_\ell}) \geq \ell.$$

In 1999, Damien Roy [2] initiated a promising approach to attack Schanuel's Conjecture. He proved that Schanuel's Conjecture is equivalent to the following algebraic statement where the symbol D stands for the derivation:

$$D = \frac{\partial}{\partial X_0} + X_1 \frac{\partial}{\partial X_1}$$

in the field $\mathbb{C}(X_0, X_1)$ and $\mathbb{Z}[X_0, X_1]_{d_0,d_1,h}$ is the set of all polynomials in $\mathbb{Z}[X_0, X_1]$ with partial degree $\leq d_0$ in X_0, partial degree $\leq d_1$ in X_1 and height $\leq h$ (the height is the maximal absolute value of the coefficients).

Conjecture 1.2 (Roy [2]). *Let ℓ be a positive integer and let $y_1, \ldots, y_\ell \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$ and $\alpha_1, \ldots, \alpha_\ell \in \mathbb{C}^\times$. Moreover, let s_0, s_1, t_0, t_1, u be positive numbers satisfying*

$$\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\}, \qquad \max\{s_0, s_1 + t_1\} < u < \frac{1}{2}(1 + t_0 + t_1). \tag{1}$$

Assume that, for any sufficiently large positive integer N, there exists a nonzero polynomial $P_N \in \mathbb{Z}[X_0, X_1]_{N^{t_0},N^{t_1},e^N}$ which satisfies

$$\Big|(D^k P_N)(\sum_{j=1}^{\ell} m_j y_j, \prod_{j=1}^{\ell} \alpha_j^{m_j})\Big| \leq \exp(-N^u)$$

for any integers $k, m_1, \ldots, m_\ell \in \mathbb{N}$ with $k \leq N^{s_0}$ and $\max\{m_1, \ldots, m_\ell\} \leq N^{s_1}$. Then

$$\mathrm{tr.deg}_{\mathbb{Q}}\mathbb{Q}(y_1, \ldots, y_\ell, \alpha_1, \ldots, \alpha_\ell) \geq \ell.$$

Damien Roy showed that if Conjecture 1.2 is true for some positive integer ℓ and some choice of parameters s_0, s_1, t_0, t_1, u satisfying (1), then Schanuel's Conjecture is true for this value of ℓ. Conversely, he showed that if Conjecture 1.1 is true for some positive integer ℓ, then Conjecture 1.2 is also true for the same value of ℓ and for any choice of parameters satisfying (1). Therefore if we would like to prove Schanuel's Conjecture according to this approach, we wish to be allowed to use parameters in a range as wide as possible.

Here we broaden the range of admissible parameters s_0, s_1, u with the following statement:

Conjecture 1.3. *Let ℓ be a positive integer and let $y_1, \ldots, y_\ell \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$ and $\alpha_1, \ldots, \alpha_\ell \in \mathbb{C}^\times$. Moreover, let s_0, s_1, t_0, t_1, u be positive numbers satisfying*

$$\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\} < u < \frac{1}{2}(1 + t_0 + t_1) \tag{2}$$

Assume that, for any sufficiently large positive integer N, there exists a nonzero polynomial $P_N \in \mathbb{Z}[X_0, X_1]_{N^{t_0},N^{t_1},e^N}$ which satisfies

$$\Big|(D^k P_N)(\sum_{j=1}^{\ell} m_j y_j, \prod_{j=1}^{\ell} \alpha_j^{m_j})\Big| \leq \exp(-N^u)$$

for any integers $k, m_1, \ldots, m_\ell \in \mathbb{N}$ *with* $k \le N^{s_0}$ *and* $\max\{m_1, \ldots, m_\ell\} \le N^{s_1}$. *Then*

$$\mathrm{tr.deg}_{\mathbb{Q}}\mathbb{Q}(y_1, \ldots, y_\ell, \alpha_1, \ldots, \alpha_\ell) \ge \ell.$$

The range of admissible pairs (t_0, t_1) in both Conjecture 1.2 and Conjecture 1.3 is the same. It consists exactly of the points in the interior of the triangle with vertices $(1/2, 1/2)$, $(1, 0)$ and $(2, 1)$. For such a point (t_0, t_1), there always exist s_0, s_1, u satisfying (1) and (2).

However, (2) of Conjecture 1.3 are really weaker than (1) of Conjecture 1.2. Indeed, with $0 < \epsilon < \dfrac{1}{8}$, choose

$$u = 1 + \epsilon, \quad s_1 = \frac{1}{2} + \epsilon, \quad t_1 = \frac{1}{2} + \frac{\epsilon}{3}, \quad t_0 = 1 + \frac{2}{3}\,\epsilon, \quad 1 + \frac{2}{3}\,\epsilon < s_0 < 1 + \epsilon,$$

then u, t_0, t_1, s_0, s_1 satisfy (2) but not (1).

Theorem 1.4. *If Conjecture 1.3 is true for some positive integer* ℓ *and some choice of parameters* s_0, s_1, t_0, t_1, u *satisfying (2), then Schanuel's Conjecture is true for this value of* ℓ. *Conversely, if Conjecture 1.1 is true for some positive integer* ℓ, *then Conjecture 1.3 is also true for the same value of* ℓ *and for any choice of parameters satisfying (2).*

Hence if Schanuel's Conjecture is true, then Conjecture 1.3 is also valid for a wider range of s_0, s_1, u. Conversely, if Conjecture 1.3 is valid for some value of the parameters which is not in the range of (1), then the result stated by Damien Roy does not allow us to deduce Schanuel's Conjecture, but Conjecture 1.3 does.

2 Equivalence between Conjecture 1.3 and Conjecture 1.1

The implication Conjecture 1.1 $\Longrightarrow$ Conjecture 1.3 follows, as in [2], from the following result which is a restatement of Proposition 3 of [2].

Theorem 2.1. *Let* $(y, \alpha) \in \mathbb{C}\times\mathbb{C}^\times$ *and let* s_0, s_1, t_0, t_1, u *be positive numbers such that*

$$\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\} < u.$$

Suppose that for any sufficiently large positive integer N, *there exists a nonzero polynomial* $P_N \in \mathbb{Z}[X_0, X_1]_{N^{t_0}, N^{t_1}, e^N}$ *which satisfies*

$$|(D^k P_N)(ny, \alpha^n)| \le \exp(-N^u)$$

for $k, n \in \mathbb{N}$ *with* $k \le N^{s_0}$ *and* $n \le N^{s_1}$. *Then there exists an integer* $d \ge 1$ *such that* $\alpha^d = e^{dy}$.

The reverse implication Conjecture 1.3 $\Longrightarrow$ Conjecture 1.1 can be deduced from the following theorem by using the appropriate choice of parameters

$$\Delta = N, \quad c = |y_1| + \cdots + |y_\ell|, \quad r = 1 + cN^{s_1}, \quad T_0 = N^{t_0}, \quad T_1 = N^{t_1}, \quad U = 2N^u.$$

Theorem 2.2. *Let* $r \ge 1, E > 1, \Delta, T_0, T_1, U$ *be positive numbers such that there exists a positive integer* K *satisfying*

$$\Delta + U + ErT_1 + T_0 \log(Er) + \log 4 < K \tag{3}$$

and

$$\left[(T_0+1)(T_1+1)\left(\frac{K}{T_1}\right)^{T_0} T_1^K e^{\Delta}\right]^K < e^{\Delta(T_0+1)(T_1+1)}. \tag{4}$$

Then there exists a non zero polynomial $P \in \mathbb{Z}[X_0, X_1]_{T_0, T_1, e^\Delta}$ *such that* $f(z) = P(z, e^z)$ *satisfies* $|f|_r \le e^{-U}$.

Proof of Theorem 2.2. From (4), applying Siegel's lemma ([4], Lemma 1.2.1), there exists $P \in \mathbb{Z}[X_0, X_1]_{T_0, T_1, e^\Delta}$ such that $f(z) = P(z, e^z)$ has a zero of multiplicity $\ge K$ at the origin. By Schwarz's lemma, we deduce

$$|f|_r \le 4.E^{-K} e^{\Delta + ErT_1} (Er)^{T_0}.$$

Finally, from (3) one deduces $|f|_r \le e^{-U}$. $\square$

3 Proof of Conjecture 1.3 for $\ell = 1$

For $\ell = 1$, Schanuel's Conjecture is just the Hermite-Lindemann Theorem: *If* α *is a nonzero complex number, one at least of the two numbers* α, e^α *is transcendental.* Therefore Conjecture 1.3 for $\ell = 1$ is also equivalent to the Hermite - Lindemann Theorem. Here we give a direct proof of the case $\ell = 1$ of Conjecture 1.3, which therefore amounts to giving a proof of the Hermite-Lindemann Theorem.

Theorem 3.1. *Let* y, α *be non zero complex numbers. Moreover, let* s_0, s_1, t_0, t_1, u *be positive numbers satisfying (2). Assume that, for any sufficiently large positive integer* N, *there exists a nonzero polynomial* $P_N \in \mathbb{Z}[X_0, X_1]_{N^{t_0}, N^{t_1}, e^N}$ *which satisfies*

$$\left|(D^k P_N)(my, \alpha^m)\right| \le \exp(-N^u) \tag{5}$$

for any integers $k, m \in \mathbb{N}$ *with* $k \le N^{s_0}$ *and* $m \le N^{s_1}$. *Then*

$$\text{tr.deg}_{\mathbb{Q}}\mathbb{Q}(y, \alpha) \geq 1.$$

Proof of Theorem 3.1. Put $F(z) = P(z, e^z)$. From Theorem 2.1 it follows that there exists a positive integer d such that $\alpha^d = e^{dy}$. Then (5) implies

$$\left| \left(\frac{d}{dz}\right)^k F_N(my) \right| < e^{-N^u}$$

for any $k, m \in \mathbb{N}$ with $0 \leq k \leq N^{s_0}$, $0 \leq m \leq N^{s_1}$ and $d|m$.

By Tijdeman's zero estimate ([3], Theorem 6.1.1) we deduce that there exist k, m such that the number

$$\left(\frac{d}{dz}\right)^k F_N(my)$$

is nonzero. From Liouville's inequalities ([4], Lemma 1.1.3) it follows that at least one of α, y is not algebraic. $\square$

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