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Functional Equation Analogous to the 2-Dimensional Wave Equation

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Abstract: In this paper, we obtained the general solution for the functional equation

$$(\Delta^2_{x,h_1} f)(x, y, t) + (\Delta^2_{y,h_2} f)(x, y, t) = (\Delta^2_{t,h_3} f)(x, y, t)$$

analogous to 2-dimensional wave equation.

Keywords: functional equation, wave equation, difference equation

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1 Introduction

In 1988, S. Haruki [2] investigated the functional equation

$$\frac{f(x+t, y) - 2f(x, y) + f(x-t, y)}{t^2} = \frac{f(x, y+s) - 2f(x, y) + f(x, y-s)}{s^2} \tag{1}$$

analogous to the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2},$$

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and obtained the general solution

$$\begin{aligned} f(x,y) = {} & c_0 + c_1(x^2+y^2) + c_2(x^3+3xy^2) + c_3(y^3+3x^2y) + c_4(x^3y+xy^3) \\ & + A_1(x) + A_2(y) + B(x,y). \end{aligned}$$

In this paper, we will find all functions $f : \mathbb{R}^3 \to \mathbb{R}$ which satisfy the functional equation

$$\begin{aligned} & \frac{f(x+h_1,y,t)-2f(x,y,t)+f(x-h_1,y,t)}{h_1^2} + \frac{f(x,y+h_2,t)-2f(x,y,t)+f(x,y-h_2,t)}{h_2^2} \\ & = \frac{f(x,y,t+h_3)-2f(x,y,t)+f(x,y,t-h_3)}{h_3^2} \end{aligned} \tag{2}$$

for all $x, y, t \in \mathbb{R}$ and $h_1, h_2, h_3 \in \mathbb{R}\setminus\{0\}$. Note that Eq.(2) can be viwed as an analogue of the 2-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial t^2}.$$

2 Preliminaries

In order to better understand the functional equation (2) and to elucidate the analogue between differential equations and functional equations, we will define the difference operator Δ_h for a function $f : \mathbb{R} \to \mathbb{R}$ by

$$\Delta_h f(x) = \frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{h}$$

for all $x \in \mathbb{R}$ and $h \in \mathbb{R}\setminus\{0\}$.

For a function $f : \mathbb{R}^3 \to \mathbb{R}$, we will define

$$\begin{aligned} \Delta_{x,h} f(x,y,t) &= \frac{f(x+\frac{h}{2},y,t) - f(x-\frac{h}{2},y,t)}{h} \\ \Delta_{y,h} f(x,y,t) &= \frac{f(x,y+\frac{h}{2},t) - f(x,y-\frac{h}{2},t)}{h} \\ \Delta_{t,h} f(x,y,t) &= \frac{f(x,y,t+\frac{h}{2}) - f(x,y,t-\frac{h}{2})}{h}. \end{aligned}$$

An iterative of the operator Δ_h is simply defined by

$$\Delta_h^n = \Delta_h(\Delta_h^{n-1}).$$

It should be noted that

$$\Delta_h^2 f(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

Now equation (1) and (2) can be written succinctly as

$$\Delta_{x,t}^2 f(x,y) = \Delta_{y,s}^2 f(x,y) \tag{3}$$

and

$$\Delta_{x,h_1}^2 f(x,y,t) + \Delta_{y,h_2}^2 f(x,y,t) = \Delta_{t,h_3}^2 f(x,y,t) \tag{4}$$

respectively.

Haruki [2] gave a remarkable lemma that will be fruitful to our work; that is, he solved the functional equation $\Delta_y^2 \psi(x) = \varphi(x)$ which simply states that the second-order difference is independent of the span.

Lemma 2.1. *(Haruki) Two functions* $\psi, \varphi : \mathbb{R} \to \mathbb{R}$ *satisfy the equation*

$$\Delta_y^2 \psi(x) = \varphi(x)$$

for all $x \in \mathbb{R}$ *and* $y \in \mathbb{R} \setminus \{0\}$ *if and only if there exists an additive function* $A : \mathbb{R} \to \mathbb{R}$ *and* $a_1, a_2, a_3 \in \mathbb{R}$ *such that*

$$\psi(x) = a_1 + A(x) + a_2 x^2 + a_3 x^3,$$
$$\varphi(x) = 2a_2 + 6a_3 x. \qquad \square$$

Please recall that an additive function $A : \mathbb{R} \to \mathbb{R}$ possesses the additive property,

$$A(x+y) = A(x) + A(y)$$

for all $x, y \in \mathbb{R}$.

Haruki applied Lemma 2.1 to the functional equation (3) and obtained the following result:

Theorem 2.2. *(Haruki) A function* $f : \mathbb{R}^2 \to \mathbb{R}$ *satisfies* $\Delta_{x,t}^2 f(x,y) = \Delta_{y,s}^2 f(x,y)$ *for all* $x, y \in \mathbb{R}$ *and* $s, t \in \mathbb{R} \setminus \{0\}$ *if and only if*

$$\begin{aligned} f(x,y) = \ & a_0 + a_1(x^2+y^2) + a_2(3x^2 y + y^3) + a_3(3xy^2 + x^3) + a_4(x^3 y + xy^3) \\ & + A_1(x) + A_2(y) + B(x,y) \end{aligned}$$

where a_0, a_1, a_2, a_3, a_4 *are constants,* A_1, A_2 *are additive functions and* B *is a bi-additive function.*

Please be reminded that a function $B:\mathbb{R}^2\to\mathbb{R}$ will be bi-additive when it is additive in each variable; that is,

$$\begin{aligned} &B(x_1+x_2,y)=B(x_1,y)+B(x_2,y)\\ \text{and}\qquad &B(x,y_1+y_2)=B(x,y_1)+B(x,y_2)\end{aligned}$$

for all $x,y\in\mathbb{R}$.

In addition, a function $f:\mathbb{R}^3\to\mathbb{R}$ will be called *3-additive* if it is additive in each variable.

3 Main result

In order to solve the functional equation

$$\Delta^2_{x,h_1}f(x,y,t)+\Delta^2_{y,h_2}f(x,y,t)=\Delta^2_{t,h_3}f(x,y,t)$$

we will first state the following lemma.

Lemma 3.1. *Let* $g:\mathbb{R}^2\to\mathbb{R}$ *and* $f:\mathbb{R}^3\to\mathbb{R}$ *be functions such that* f *is additive in the first variable. Then* f *and* g *will satisfy the following system of equations*

$$\begin{aligned}\Delta^2_{y,h_1}g(y,t)&=\Delta^2_{t,h_2}g(y,t)\\ xg(y,t)+\Delta^2_{y,h_1}f(x,y,t)&=\Delta^2_{t,h_2}f(x,y,t)\end{aligned}\tag{5}$$

if and only if

$$\begin{aligned}f(x,y,t)=\;&C_0(x)+(y^2+t^2)C_1(x)+(t^3+3y^2t)C_2(x)+(y^3+3yt^2)C_3(x)\\ &+(yt^3+y^3t)C_4(x)+b_0xt^2+A_1(x,t)+A_2(x,y)+A_3(x,y,t)\\ &+xt^2B_1(y)+xt^3B_3(y)-xy^2B_2(t)-xy^3B_4(t)\end{aligned}\tag{6}$$

$$g(y,t)=\;2b_0+2B_1(y)+2B_2(t)+6tB_3(y)+6yB_4(t)\tag{7}$$

where B_1,B_2,B_3,B_4 *and* C_1,C_2,C_3,C_4 *are additive functions,* A_1,A_2 *are bi-additive,* A_3 *is 3-additive and* b_0 *is a constant.*

Proof. By Theorem 2.2, the function g is given by

$$\begin{aligned}g(y,t)=\;&b_0+b_1(y^2+t^2)+b_2(y^3+3yt^2)+b_3(t^3+3y^2t)+b_4(y^3t+yt^3)\\ &+B_1(y)+B_2(t)+B^*(y,t),\end{aligned}\tag{8}$$

where $b_0, b_1, \ldots, b_4$ are constants, B_1, B_2 are additive and B^* is bi-additive. From Eq.(5), we can see that $\Delta^2_{y,h_1} f(x,y,t) = \Delta^2_{t,h_2} f(x,y,t) - xg(y,t)$ which is independent of the span h_1. Thus, by Lemma 2.1, we have

$$f(x,y,t) = D_0(x,t) + D_1(x,y,t) + y^2 D_2(x,t) + y^3 D_3(x,t), \tag{9}$$

where D_1 is additive in the second variable. Substitute Eq.(8) and Eq.(9) back into Eq.(5),

$$\begin{aligned} x\Big(b_0 + b_1(y^2+t^2) + b_2(y^3+3yt^2) + b_3(t^3+3y^2t) + b_4(y^3t+yt^3) \\ +B_1(y) + B_2(t) + B^*(y,t)\Big) + 2D_2(x,t) + 6yD_3(x,t) \\ = \Delta^2_{t,h_3}(D_0(x,t) + D_1(x,y,t) + y^2D_2(x,t) + y^3D_3(x,t)). \end{aligned} \tag{10}$$

Observe that for arbitrary $r \in \mathbb{Q}$, substituting ry for y in Eq.(10), we obtain a polynomial of variable r with all rational numbers being its roots. Hence all the coefficients of the polynomial (in terms of the variable r) must vanish, that is,

$$b_0x + b_1xt^2 + b_3xt^3 + xB_2(t) + 2D_2(x,t) = \Delta^2_{t,h_2} D_0(x,t), \tag{11}$$

$$3b_2xyt^2 + b_4xyt^3 + xB_1(y) + xB^*(y,t) + 6yD_3(x,t) = \Delta^2_{t,h_2} D_1(x,y,t), \tag{12}$$

$$b_1xy^2 + 3b_3xy^2t = \Delta^2_{t,h_2} y^2 D_2(x,t), \tag{13}$$

$$b_2xy^3 + b_4xy^3t = \Delta^2_{t,h_2} y^3 D_3(x,t), \tag{14}$$

From Eq.(13), using Lemma 2.1, we will have

$$D_2(x,t) = C_1(x) + E_1(x,t) + \frac{b_1}{2}xt^2 + \frac{b_3}{2}xt^3, \tag{15}$$

where E_1 is additive in the second variable. Substitute Eq.(15) into Eq.(11) to get

$$\Delta^2_{t,h_2} D_0(x,t) = b_0x + 2C_1(x) + xB_2(t) + 2E_1(x,t) + 2b_1xt^2 + 2b_3xt^3. \tag{16}$$

By Lemma 2.1, the right-hand side of (16) must be a polynomial of degree 1 in the variable t. Therefore, $b_1 = 0 = b_3$ and $xB_2(t) + 2E_1(x,t) = tC_2(x)$ for some $C_2 : \mathbb{R} \to \mathbb{R}$. Moreover, D_0 must be of the form,

$$D_0(x,t) = C_0(x) + A_1(x,t) + \frac{b_0x + 2C_1(x)}{2}t^2 + \frac{C_2(x)}{6}t^3 \tag{17}$$

where A_1 is additive in the second variable. Since $b_1 = 0 = b_3$, Eq.(15) becomes

$$D_2(x,t) = C_1(x) + E_1(x,t) = C_1(x) + \frac{tC_2(x) - xB_2(t)}{2}. \tag{18}$$

Similarly, from Eq.(14) and Lemma 2.1, we get

$$D_3(x,t) = C_3(x) + E_2(x,t) + \frac{b_2}{2}xt^2 + \frac{b_4}{6}xt^3, \tag{19}$$

with E_1 additive in the second variable. By Eq.(12) and (19), we obtain

$$\Delta^2_{t,h_2} D_1(x,y,t) = xB_1(y) + 6yC_3(x) + xB^*(y,t) + 6yE_2(x,t) + 6b_2xyt^2 + 2b_4xyt^3$$

By Lemma 2.1, as before, $b_2 = 0 = b_4$, $xB^*(y,t) + 6yE_2(x,t) = tE_3(x,y)$ and

$$D_1(x,y,t) = A_2(x,y) + A_3(x,y,t) + \frac{xB_1(y) + 6yC_3(x)}{2}t^2 + \frac{E_3(x,y)}{6}t^3 \tag{20}$$

and Eq.(19) becomes

$$D_3(x,t) = C_3(x) + E_2(x,t) \tag{21}$$

where A_3 is additive in the third variable. Note that E_3 is additive in the second variable since $E_3(x,y) = xB^*(y,1) + 6yE_2(x,1)$. If we substitute $t = 0$ into Eq.(20), we can see that $A_2(x,y) = D_1(x,y,0)$ which is additive in the second variable. From Eq.(20), all functions other that A_3 are additive in the second variable. Hence, A_3 must also be additive in the second variable.

Gathering all we have so far and substituting Eqs.(17), (18), (20) and (21) into Eq.(9)

$$\begin{aligned}
f(x,y,t) =\ & C_0(x) + A_1(x,t) + \frac{b_0x + 2C_1(x)}{2}t^2 + \frac{C_2(x)}{6}t^3 \\
& + A_2(x,y) + A_3(x,y,t) + \frac{xB_1(y) + 6yC_3(x)}{2}t^2 + \frac{E_3(x,y)}{6}t^3 \\
& + y^2(C_1(x) + \frac{tC_2(x) - xB_2(t)}{2}) + y^3(C_3(x) + E_2(x,t)) \\
=\ & C_0(x) + (y^2+t^2)C_1(x) + (t^3 + 3y^2t)\frac{C_2(x)}{6} + (y^3 + 3yt^2)C_3(x) \\
& + \frac{b_0}{2}xt^2 + A_1(x,t) + A_2(x,y) + A_3(x,y,t) \\
& + xt^2\frac{B_1(y)}{2} + t^3\frac{E_3(x,y)}{6} - xy^2\frac{B_2(t)}{2} + y^3E_2(x,t) \qquad (22) \\
g(y,t) =\ & b_0 + B_1(y) + B_2(t) + tE_3(1,y) - 6yE_2(1,t) \qquad (23)
\end{aligned}$$

Now that we have obtained g, we will show that each function in Eq.(22) is additive in the first variable. Since $C_0(x) = f(x,0,0)$, we have that C_0 is additive. If we substitute $t = 0$ into Eq.(22), we have that

$$\varphi_y(x) \equiv y^2C_1(x) + y^3C_3(x) + A_2(x,y) = f(x,y,0) - C_0(x)$$

where we have defined $\varphi_y(x)$ to be the term on the left-hand side of the above equation. Since $f(x,y,0)-C_0(x)$ is an additive function of x, $\varphi_y(x)$ is also an additive function of x. One can verify that

$$\begin{aligned} C_1(x) &= -\frac{5}{2}\varphi_1(x)+2\varphi_2(x)-\frac{1}{2}\varphi_3(x),\\ C_3(x) &= \frac{1}{2}\varphi_1(x)-\frac{1}{2}\varphi_2(x)+\frac{1}{6}\varphi_3(x),\\ A_2(x,y) &= 3\varphi_y(x)-\frac{3}{2}\varphi_{2y}(x)+\frac{1}{3}\varphi_{3y}(x). \end{aligned}$$

Hence C_1 and C_3 are additive and A_2 is bi-additive (note that A_2 is already additive in the second variable). For other functions in Eq. (22), we can show in a similar way that each of them is additive in the first variable.

Now we substitute Eq.(22) and Eq.(23) into Eq.(5), we get

$$tE_3(x,y)-xtE_3(1,y)=6yE_2(x,t)-6xyE_2(1,t). \tag{24}$$

Define $T(x,y)=E_3(x,y)-xE_3(1,y)$. By Eq.(24), we have

$$T(x,y)=6yE_2(x,1)-6xyE_2(1,1)=yC_4(x),$$

where $C_4(x)=6E_2(x,1)-6xE_2(1,1)$. Note that C_4 is additive. From the definition of T, we get

$$tE_3(x,y)=xtE_3(1,y)+ytC_4(x). \tag{25}$$

From Eqs.(24) and (25), we obtain

$$6yE_2(x,t)=6xyE_2(1,t)+ytC_4(x). \tag{26}$$

Substituting Eqs.(25) and (26) into Eq.(22), we get the functional equation f and g as in Eqs.(6) and (7).

Conversely, if f and g are given by Eqs.(6) and (7), respectively, then it can be verified that (5) holds. This completes the proof. □

Corollary 3.2. *Let $f,g:\mathbb{R}^2\to\mathbb{R}$ such that g satisfies Eq.(3). Then*

$$g(y,t)+\Delta^2_{y,h_1}f(y,t)=\Delta^2_{t,h_2}f(y,t) \tag{27}$$

if and only if

$$\begin{aligned} f(y,t) &= a_0 + a_1(y^2+t^2) + a_2(t^3+3y^2t) + a_3(y^3+3yt^2) \\ &\quad + a_4(yt^3+y^3t) + b_0t^2 + A_1(t) + A_2(y) + A_3(y,t) \\ &\quad + t^2B_1(y) + t^3B_3(y) - y^2B_2(t) - y^3B_4(t) \\ g(y,t) &= 2b_0 + 2B_1(y) + 2B_2(t) + 6tB_3(y) + 6yB_4(t) \end{aligned}$$

where $a_0, a_1, \ldots, a_4, b_0$ *are constants,* $A_1, A_2, B_1, B_2, B_3, B_4$ *are additive functions and* A_3 *is 3-additive.*

Proof. Multiplying Eq.(27) by x, we get

$$xg(y,t) + \Delta^2_{y,h_1} xf(y,t) = \Delta^2_{t,h_2} xf(y,t).$$

Define $f^*(x,y,t) = xf(y,t)$. The above equation then becomes

$$xg(y,t) + \Delta^2_{y,h_1} f^*(x,y,t) = \Delta^2_{t,h_2} f^*(x,y,t)$$

Applying Lemma 3.1 and using the fact that $f(y,t) = f^*(1,y,t)$, we get the desired conclusion. □

Now that we have Lemma 3.1, we are ready to solve Eq.(4).

Theorem 3.3. *A function* $f : \mathbb{R}^3 \to \mathbb{R}$ *satisfy (4) if and only if*

$$\begin{aligned} f(x,y,t) = \ & a_0 + (A_1(x)+a_1)(y^2+t^2) + (A_2(x)+a_2)(t^3+3y^2t) \\ & + (A_3(x)+a_3)(y^3+3yt^2) + (A_4(x)+a_4)(y^3t+yt^3) \\ & + (A_5(y)+a_5)(x^2+t^2) + (A_6(y)+a_6)(x^3+3xt^2) \\ & + (A_7(y)+a_7)(t^3+3x^2t) + (A_8(y)+a_8)(x^3t+xt^3) \\ & + (A_9(t)+a_9)(x^2-y^2) + (A_{10}(t)+a_{10})(y^3-3x^2y) \\ & + (A_{11}(t)+a_{11})(x^3y-xy^3) + (A_{12}(t)+a_{12})(x^3-3xy^2) + A_{13}(x) \\ & + A_{14}(t) + A_{15}(y) + B_1(x,y) + B_2(y,t) + B_3(x,t) + T_3(x,y,t) \end{aligned}$$

where $a_1, a_2, \ldots, a_{12}$ *are constants,* $A_1, A_2, \ldots, A_{15}$ *are additive,* B_1, B_2, B_3 *'s are bi-additive and* T *is 3-additive.*

Proof. Firstly, we put $h_2 = 1 = h_3$ in Eq.(4) and then apply Lemma 2.1. We obtain

$$f(x,y,t) = A(y,t) + B(x,y,t) + x^2C(y,t) + x^3D(y,t) \tag{28}$$

where B is additive in the first variable. Substitute Eq.(28) into Eq.(4), we have

$$\begin{aligned}2C(y,t)+6xD(y,t)+\Delta^2_{y,h_2}(A(y,t)+B(x,y,t)+x^2C(y,t)+x^3D(y,t))\\ =\Delta^2_{t,h_3}(A(y,t)+B(x,y,t)+x^2C(y,t)+x^3D(y,t)).\end{aligned}$$

If we replace x with rx, where $r\in\mathbb{Q}$, we will get a polynomial of r with infinite number of roots, and hence all of its coefficients must be zero;

$$2C(y,t)+\Delta^2_{y,h_2}A(y,t)=\Delta^2_{t,h_3}A(y,t), \tag{29}$$

$$6xD(y,t)+\Delta^2_{y,h_2}B(x,y,t)=\Delta^2_{t,h_3}B(x,y,t), \tag{30}$$

$$x^2\Delta^2_{y,h_2}C(y,t)=x^2\Delta^2_{t,h_3}C(y,t), \tag{31}$$

$$x^3\Delta^2_{y,h_2}D(y,t)=x^3\Delta^2_{t,h_3}D(y,t). \tag{32}$$

Applying Corollary 3.2 to Eq.(29) and Eq.(31), we have

$$\begin{aligned}A(y,t)=\ & a_0+a_1(y^2+t^2)+a_2(t^3+3y^2t)+a_3(y^3+3yt^2)+a_4(y^3t+yt^3)+c_0t^2\\ &+A_1(t)+A_2(y)+A_3(y,t)+t^2C_1(y)+t^3C_3(y)-y^2C_2(t)-y^3C_4(t)\\ C(y,t)=\ & c_0+C_1(y)+C_2(t)+3tC_3(y)+3yC_4(t)\end{aligned}$$

and when applying Lemma 3.1 to Eq.(30) and Eq.(32), we obtain

$$\begin{aligned}B(x,y,t)=\ & B_0(x)+(y^2+t^2)B_1(x)+(t^3+3y^2t)B_2(x)+(y^3+3yt^2)B_3(x),\\ &+(y^3t+yt^3)B_4(x)+3d_0xt^2+E_1(x,t)+E_2(x,y)+E_3(x,y,t),\\ &+3xt^2D_1(y)+3xt^3D_3(y)-3xy^2D_2(t)-3xy^3D_4(t),\\ D(y,t)=\ & d_0+D_1(y)+D_2(t)+3tD_3(y)+3yD_4(t).\end{aligned} \tag{33}$$

Now we have

$$\begin{aligned}f(x,y,t)=\ & a_0+(B_1(x)+a_1^*)(y^2+t^2)+(B_2^*(x)+a_2^*)(t^3+3y^2t)\\ &+(B_3(x)+a_3^*)(y^3+3yt^2)+(B_4(x)+a_4)(y^3t+yt^3)\\ &+(C_1^*(y)+c_0^*)(x^2+t^2)+(D_1(y)+d_0^*)(x^3+3xt^2)\\ &+(C_3(y)+k_1)(t^3+3x^2t)+(3D_3(y)+k_2)(x^3t+xt^3)\\ &+(C_2^*(t)+k_3)(x^2-y^2)-(C_4(t)+k_4)(y^3-3x^2y)\\ &+(D_2^*(t)+k_5)(x^3-3xy^2)+(3D_4(t)+k_6)(x^3y-xy^3)\\ &+B_0(x)+A_1(t)+A_2(y)+E_2(x,y)+A_3(y,t)+E_1(x,t)+E_3(x,y,t)\end{aligned}$$

where we have defined

$$C_2(t)^* = C_2(t) - 3k_1(t), \qquad a_2^* = a_2 - k_1,$$
$$D_2^*(t) = D_2(t) - k_2 t, \qquad B_2^*(x) = B_2(x) - k_2 x,$$
$$c_0^* = c_0 - k_3 \qquad a_1^*, = a_1 + k_3,$$
$$a_3^* = a_3 + k_4 + k_6, \qquad C_1^*(y) = C_1(y) - 3k_4 y,$$
$$d_0^* = d_0 - k_5 - k_6, \qquad B_1^*(x) = B_1(x) + 3k_5 x.$$

It is straightforward to verify that the function of the above form is indeed the general solution of Eq.(4) □

The next corollary extends our result to a difference functional equation analogous to the wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

Corollary 3.4. *Let* $f_1, f_2 : \mathbb{R}^3 \to \mathbb{R}$ *and* $c \in \mathbb{R} \setminus \{0\}$ *such that* $f_2(x,y,t) = f_1(x,y,ct)$*. Then* f_1 *satisfies Eq.(4) if and only if* f_2 *satisfies the equation*

$$c^2(\Delta^2_{x,h_1} f(x,y,t) + \Delta^2_{y,h_2} f(x,y,t)) = \Delta^2_{t,h_3} f(x,y,t). \tag{34}$$

Proof. Observe that

$$\begin{aligned} & c^2((\Delta^2_{1,h_1} f_2)(x,y,t) + (\Delta^2_{2,h_2} f_2)(x,y,t)) - (\Delta^2_{3,h_3} f_2)(x,y,t) \\ = \ & c^2((\Delta^2_{1,h_1} f_2)(x,y,t) + (\Delta^2_{2,h_2} f_2)(x,y,t)) \\ & - (\frac{f_2(x,y,t+h_3) - 2f_2(x,y,t) + f_2(x,y,t-h_3)}{h_3^2}) \\ = \ & c^2((\Delta^2_{1,h_1} f_1)(x,y,ct) + (\Delta^2_{2,h_2} f_1)(x,y,ct)) \\ & - c^2(\frac{f_1(x,y,ct+ch_3) - 2f_1(x,y,ct) + f_1(x,y,ct-ch_3)}{(ch_3)^2}) \\ = \ & c^2((\Delta^2_{1,h_1} f_1)(x,y,t') + (\Delta^2_{2,h_2} f_1)(x,y,t')) - c^2(\Delta^2_{3,h_3'} f_1)(x,y,t') \end{aligned}$$

where $t' = ct$ and $h_3' = ch_3$. Hence f_1 satisfies Eq.(4) if and only if f_2 satisfies Eq.(34). □

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