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Module Amenability of Banach Algebras

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Abstract: Let $\mathfrak{A}$ and $\mathcal{A}$ be Banach algebras, and let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule. In this paper, at first we generalize some theorems from amenable Banach algebras into module amenable Banach algebras. We show that when $\mathcal{A}$ and I are commutative Banach $\mathfrak{A}$-bimodules, and $\mathcal{A}$ is module amenable, where I is two-sided closed ideal in $\mathcal{A}$, then I is module amenable. By this, we show that if I is a two sided ideal in an amenable inverse semigroup S, then I is amenable.

Keywords: Amenability, Banach algebras, Module amenability, Semigroup, Semigroup algebras

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1 Introduction

The concept of amenability for Banach algebras was introduced by Johnson in 1972 [8]. The Banach algebra $\mathcal{A}$ is said to be amenable if $H^1(\mathcal{A}, \mathcal{X}^*) = \{0\}$ for all Banach A-bimodule $\mathcal{X}$, such that $\mathcal{X}^*$ is the first dual of $\mathcal{X}$.

In [1], Amini introduced the concept of module amenability of Banach algebras. He showed that under some natural conditions, for an inverse semigroup S with the set of idempotents E_S, $\ell^1(S)$ is $\ell^1(E_S)$-module amenable if and only if S is amenable. Amini and Bodaghi studied this version of amenability in [2].

For an amenable Banach algebra $\mathcal{A}$, every closed ideal I is amenable if and only if I has a bounded approximate identity if and only if I is weakly complemented in $\mathcal{A}$ (Theorem 2.3.7 of [13]). Zhang in [14], showed that if I is approximately complemented in $\mathcal{A}$, then the above results are hold.

In the next section, we prove a similar Theorem to Theorem 2.3.7 of [13] for module amenability of Banach algebras. By this Theorem we prove that if S be an amenable inverse semigroup, and I is an ideal in S, then I is also amenable. This proof is different from to prove of Corollary 1.22 of [10].

2 Module Amenability of Banach Algebras

Let $\mathfrak{A}$ and $\mathcal{A}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodules with following compatible actions

$$\alpha.(ab) = (\alpha.a)b, \quad a(\alpha.b) = (a.\alpha)b, \tag{2.1}$$

and

$$(ab).\alpha = a(b.\alpha), \quad (a.\alpha)b = a(\alpha.b), \tag{2.2}$$

for every $a, b \in \mathcal{A}, \alpha \in \mathfrak{A}$. Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-bimodules with compatible actions. An $\mathfrak{A}$-module map is a mapping $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ with following properties

1. $\varphi(a \pm b) = \varphi(a) \pm \varphi(b)$;
2. $\varphi(\alpha.a) = \alpha.\varphi(a)$;
3. $\varphi(a.\alpha) = \varphi(a).\alpha$,

for every $a, b \in \mathcal{A}, \alpha \in \mathfrak{A}$. Note that φ is not linear. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with following compatible actions

$$\alpha.(a.x) = (\alpha.a).x, \quad a(\alpha.x) = (a.\alpha).x, \quad (\alpha.x).a = \alpha.(x.a), \tag{2.3}$$

and

$$(a.x).\alpha = a.(x.\alpha), \quad (a.\alpha).x = a.(\alpha.x), \quad (x.a).\alpha = x.(a.\alpha), \tag{2.4}$$

for every $x \in \mathcal{X}$, $a \in \mathcal{A}, \alpha \in \mathfrak{A}$. Then by this actions $\mathcal{X}$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule. If $\alpha.x = x.\alpha$, for every $x \in \mathcal{X}$ and $\alpha \in \mathfrak{A}$, then $\mathcal{X}$ is called a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule. Moreover, if $a.x = x.a$, for every $x \in \mathcal{X}$ and $\alpha \in \mathcal{A}$, then $\mathcal{X}$ is called a bi-commutative Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule. It is clear that $\mathcal{A}$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule. Also if $\mathcal{A}$ is a commutative $\mathfrak{A}$-bimodule, then $\mathcal{A}$ is a bi-commutative $\mathcal{A}$-$\mathfrak{A}$-bimodule. Similarly, dual, second dual and n-dual of $\mathcal{A}$ are commutative or bi-commutative $\mathcal{A}$-$\mathfrak{A}$-bimodules. $\mathcal{X}$ is called $\mathcal{A}$-essential if $\mathcal{X}\mathcal{A}\mathcal{X} = \mathcal{X}$.

An $\mathfrak{A}$-module map $D : \mathcal{A} \longrightarrow \mathcal{X}$ is called a module derivation if

$$D(ab) = a.D(b) + D(a).b \qquad (a.b \in \mathcal{A}). \tag{2.5}$$

The module derivation D is called bounded if there exists $M > 0$ such that $\|D(a)\| \leq M\|a\|$, for every $a \in \mathcal{A}$. Note that boundedness of D implies its norm continuity.

Definition 2.1. The Banach algebra $\mathcal{A}$ is called module amenable (as an $\mathfrak{A}$-bimodule) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule $\mathcal{X}$, each module derivation $D : \mathcal{A} \longrightarrow \mathcal{X}^*$ is inner.

Similarly to amenability, we use the notations $Z^1_{\mathfrak{A}}(\mathcal{A}, \mathcal{X}^*)$ for the set of all module derivations $D : \mathcal{A} \longrightarrow \mathcal{X}^*$, and $N^1_{\mathfrak{A}}(\mathcal{A}, \mathcal{X}^*)$ for those which are inner. We consider the quotient space $H^1_{\mathfrak{A}}(\mathcal{A}, \mathcal{X}^*) = Z^1_{\mathfrak{A}}(\mathcal{A}, \mathcal{X}^*)/N^1_{\mathfrak{A}}(\mathcal{A}, \mathcal{X}^*)$ called the first relative (to $\mathfrak{A}$) Hochschild cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{X}^*$. Hence $\mathcal{A}$ is module amenable if and only if $H^1_{\mathfrak{A}}(\mathcal{A}, \mathcal{X}^*) = Z^1_{\mathfrak{A}}(\mathcal{A}, \mathcal{X}^*)/N^1_{\mathfrak{A}}(\mathcal{A}, \mathcal{X}^*) = \{0\}$, for each commutative Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule $\mathcal{X}$.

Let $\mathcal{A}$ and $\mathcal{X}$ be Banach algebras; let $\mathcal{A}$ be a commutative Banach $\mathfrak{A}$-bimodule, and let $\mathcal{X}$ be a Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule. If I is a left ideal in $\mathcal{A}$, such that I is a commutative Banach $\mathfrak{A}$-bimodule, and $D : I \longrightarrow \mathcal{X}$ is a module derivation. Then for each $a \in I$, the map

$$D_a : x \mapsto D(ax) - a.Dx, \qquad I \longrightarrow \mathcal{X}, \tag{2.6}$$

is a right I-module homomorphism (Proposition 1.8.3 of [4]), and clearly is a $\mathfrak{A}$-module map. A left (right) multiplier on $\mathcal{A}$ is an element L (or R) in $\mathcal{L}(\mathcal{A})$ such that $L(ab) = L(a)b$ ($R(ab) = aR(b)$), for each $a, b \in \mathcal{A}$. A multiplier is a pair (L, R), where L and R are left and right multipliers on $\mathcal{A}$, respectively, and

$$aL(b) = R(a)b \qquad (a, b \in \mathcal{A})$$

The sets of left multipliers, right multipliers, and multipliers on $\mathcal{A}$ are denoted by $\mathcal{M}l(\mathcal{A})$, $\mathcal{M}_r(\mathcal{A})$, and $\mathcal{M}(\mathcal{A})$, respectively. They are subalgebras of $\mathcal{L}(\mathcal{A})$, $\mathcal{L}(\mathcal{A})^{op}$, and $\mathcal{L}(\mathcal{A}) \times \mathcal{L}(\mathcal{A})^{op}$, respectively.

Suppose that $\mathcal{A}$ is an ideal in a Banach algebra $\mathcal{B}$, and $b \in \mathcal{B}$. The map $\theta : \mathcal{B} \longrightarrow \mathcal{M}(\mathcal{A})$ defined by $\theta(b) = (L_b, R_b)$ is a homomorphism, where $L_b : a \mapsto ba$, and $R_b : a \mapsto ab$ on $\mathcal{A}$. This homomorphism is called regular homomorphism (for more details see p. 60 of [4]). It is clear that if both $\mathcal{A}$ and $\mathcal{B}$ are commutative Banach $\mathfrak{A}$-bimodule, then θ is a $\mathfrak{A}$-module map.

Now let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule with a bounded approximate identity, let $\mathcal{X}$ be an $\mathcal{A}$-essential, and commutative $\mathfrak{A}$-bimodule. Then $\mathcal{X}$ by following module actions

$$(L,R).(a.x) = La.x, \quad (x.a).(L,R) = x.Ra \qquad (a \in \mathcal{A}, (L,R) \in \mathcal{M}(\mathcal{A}), x \in \mathcal{X}) \tag{2.7}$$

is a unital Banach $\mathcal{M}(\mathcal{A})$-bimodule (Theorem 2.9.51 of [4]). Also we have

$$x.La = (x.\mu).a, \qquad Ra.x = a.(\mu.x), \tag{2.8}$$

for each $a \in \mathcal{A}$, $\mu = (L,R) \in \mathcal{M}(\mathcal{A})$, $x \in \mathcal{X}$. By easy argument $\mathcal{M}(\mathcal{A})$ is a $\mathfrak{A}$-bimodule.

Theorem 2.2. *Let $\mathcal{A}$ be a commutative Banach $\mathfrak{A}$-bimodule, let $\mathcal{X}$ be a $\mathcal{A}$-essential module and a commutative $\mathfrak{A}$-bimodule. Suppose that $D : \mathcal{A} \longrightarrow \mathcal{X}^*$ is a module derivation, then there is a unique module derivation $\widetilde{D} : \mathcal{M}(\mathcal{A}) \longrightarrow \mathcal{X}^*$ such that $\widetilde{D}|_{\mathcal{A}} = D$. If D is inner then $\widetilde{D}$ is also inner. Moreover, if D is bounded then $\widetilde{D}$ is also bounded.*

Proof. Let (e_α) be a bounded approximate identity for $\mathcal{A}$. By (2.7) $\mathcal{X}$, and hence $\mathcal{X}^*$, are unital $\mathcal{A}$-bimodules. Take $\mu = (L,R) \in \mathcal{M}(\mathcal{A})$, and define

$$D_\mu : a \mapsto D(\mu.a) - \mu.D(a), \qquad \mathcal{A} \longrightarrow \mathcal{X}^*. \tag{2.9}$$

By (2.6), D_μ is a right $\mathcal{A}$-module homomorphism, and so it is continuous (Theorem 2.9.30 (ix) of [4]). Therefore the bounded net $(D_\mu e_\alpha)$ has a accumulation point, λ, in w-topology. Since $\mathcal{X}$ is essential in $\mathcal{A}$, so take $x = a.y \in \mathcal{X}$, where $a \in \mathcal{A}$ and $y \in \mathcal{X}$. Then

$$\langle x, D_\mu e_\alpha \rangle = \langle y, D_\mu e_\alpha.a \rangle = \langle y, D_\mu(e_\alpha.a) \rangle \longrightarrow \langle y, D_\mu a \rangle,$$

and so $\langle x, \lambda \rangle = \langle y, D_\mu a \rangle$. Note that λ is independent of the choice of the bounded approximate identity. Set $\widetilde{D}\mu = \lambda$. Then we have

$$\langle y, D_\mu a \rangle = \langle a.y, \widetilde{D}_\mu \rangle = \langle y, \widetilde{D}\mu.a \rangle \qquad (y \in \mathcal{X}, a \in \mathcal{A}),$$

therefore we can write

$$\widetilde{D}\mu.a = D_\mu a = D(\mu.a) - \mu.Da \qquad (a \in \mathcal{A}).$$

Since $\mathcal{A}$ is a commutative $\mathfrak{A}$-bimodule, and $\mathcal{M}(\mathcal{A})$ is a $\mathfrak{A}$-bimodule, so for every $\gamma \in \mathfrak{A}$ we have

$$\begin{aligned} \widetilde{D}(\gamma.\mu).a &= D_{\gamma.\mu}a = D(\gamma.\mu.a) - \gamma.\mu.Da \\ &= \gamma.\widetilde{D}\mu.a, \end{aligned}$$

and

$$\begin{aligned} \widetilde{D}(\mu.\gamma).a &= D_{\mu.\gamma}a = D(\mu.\gamma.a) - \mu.\gamma.Da \\ &= \widetilde{D}\mu.a.\gamma. \end{aligned}$$

Therefore $\widetilde{D}$ is an $\mathfrak{A}$-module map. Let $\mu_1 = (L_1, R_1)$ and $\mu_2 = (L_2, R_2)$ in $\mathcal{M}(\mathcal{A})$. Then for each $a \in \mathcal{A}$ we have

$$\begin{aligned} \widetilde{D}(\mu_1\mu_2).a &= D_{\mu_1\mu_2}a = D(\mu_1\mu_2.a) - \mu_1\mu_2.D(a) \\ &= D(\mu_1.L_2a) - \mu_1\mu_2.D(a) = D_{\mu_1}(L_2a) + \mu_1.D_{\mu_2}a \\ &= D_{\mu_1}(\mu_2.a) + \mu_1.D_{\mu_2}a = \widetilde{D}\mu_1.\mu_2.a + \mu_1.\widetilde{D}\mu_2.a \\ &= (\widetilde{D}\mu_1.\mu_2 + \mu_1.\widetilde{D}\mu_2).a. \end{aligned}$$

So, by $\mathcal{X} = \mathcal{X}\mathcal{A}\mathcal{X}$, we have $\widetilde{D}(\mu_1\mu_2) = \widetilde{D}\mu_1.\mu_2 + \mu_1.\widetilde{D}\mu_2$, and since $\widetilde{D}$ is an $\mathfrak{A}$-module map, hence $\widetilde{D}$ is a module derivation. Let $a \in \mathcal{A}$, and set $\mu = (L_a, R_a)$. Then

$$\widetilde{D}\mu.b = D_\mu b = D(ab) - a.Db = Da.b \qquad (b \in \mathcal{A}).$$

Thus $\widetilde{D}\mu = Da$, and this means $\widetilde{D}|_{\mathcal{A}} = D$. By existing of bounded approximate identity (e_α) in $\mathcal{A}$, we can show that $\widetilde{D}$ is unique and it is easy.

Suppose that $D : \mathcal{A} \longrightarrow \mathcal{X}^*$ is an inner module derivation. Therefore there exists $\lambda \in \mathcal{X}^*$ such that $D(a) = a.\lambda - \lambda.a$, for each $a \in \mathcal{A}$. Then the inner module derivation

$$\mu \mapsto \mu.\lambda - \lambda.\mu, \qquad \mathcal{M}(\mathcal{A}) \longrightarrow \mathcal{X}^*$$

is a module derivation, which extends D. Since the extend of D is unique, hence $\widetilde{D}\mu = \mu.\lambda - \lambda.\mu$, for each $\mu \in \mathcal{M}(\mathcal{A})$.

If we suppose that D is a bounded module derivation, then there is a $M > 0$ such that $\|Da\| \leq M\|a\|$. Then for every $a \in \mathcal{A}$ and $\mu \in \mathcal{M}(\mathcal{A})$ we have

$$\|\widetilde{D}\mu.a\| = \|D_\mu a\| \leq 2\|D\|\|a\|\|\mu\| \leq 2M\|a\|\|\mu\|.$$

Thus $\widetilde{D}$ is bounded. $\square$

We denote the space of all bounded $\mathfrak{A}$-module maps from G into F by $\mathcal{B}_{\mathfrak{A}}(G,F)$, where F and G are commutative Banach $\mathfrak{A}$-bimodules. Now, Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule, and Let G and F be Banach left (right) $\mathcal{A}$-modules, and commutative Banach $\mathfrak{A}$-bimodules. By ${}_{\mathcal{A}}\mathcal{B}_{\mathfrak{A}}(G,F)$ ($\mathcal{B}_{\mathcal{A},\mathfrak{A}}(G,F)$) we mean all bounded left (right) $\mathcal{A}$-module homomorphisms, and $\mathfrak{A}$-module homomorphisms from G into F. It is clear that $\mathcal{B}_{\mathfrak{A}}(G,F)$, ${}_{\mathcal{A}}\mathcal{B}_{\mathfrak{A}}(G,F)$ and $\mathcal{B}_{\mathcal{A},\mathfrak{A}}(G,F)$ are Banach $\mathcal{A}$-bimodules.

Consider the following short exact sequence of Banach left $\mathcal{A}$-modules, and commutative Banach $\mathfrak{A}$-bimodules:

$$\sum : 0 \longrightarrow E \stackrel{S}{\longrightarrow} F \stackrel{T}{\longrightarrow} G \longrightarrow 0.$$

Where S and T are in ${}_{\mathcal{A}}\mathcal{B}_{\mathfrak{A}}(G,F)$. Then, $\sum$ is splits strongly if and only if T is a retraction (there is a $Q \in {}_{\mathcal{A}}\mathcal{B}_{\mathfrak{A}}(F,G)$ such that $T \circ Q = id_G$).

Proposition 2.3. *Let $\mathcal{A}$ be a module amenable Banach algebra, let E be a Banach right $\mathcal{A}$-module, and commutative Banach $\mathfrak{A}$-bimodule. Let F and G be Banach left $\mathcal{A}$-modules, and commutative Banach $\mathfrak{A}$-bimodules. Then each admissible short exact sequence of commutative Banach $\mathcal{A}$-$\mathfrak{A}$-bimodules*

$$\sum : 0 \longrightarrow E^* \stackrel{S}{\longrightarrow} F \stackrel{T}{\longrightarrow} G \longrightarrow 0$$

splits strongly.

Proof. Since $\sum$ is admissible, then there is a $Q_1 \in \mathcal{B}_{\mathfrak{A}}(G,F)$ with $T \circ Q_1 = id_G$. Since $\mathcal{B}_{\mathfrak{A}}(G,F)$ is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule. Define $D : \mathcal{A} \longrightarrow \mathcal{B}_{\mathfrak{A}}(G,F)$ with $D(a) = a.Q_1 - Q_1.a$. Thus D is a bounded module derivation, and for each $a \in \mathcal{A}$ and $z \in G$ we have

$$\begin{aligned} (T \circ Da)(z) &= T(a.Q_1 z - Q_1(a.z)) \\ &= a.(T \circ Q_1)(z) - (T \circ Q_1)(a.z) = a.z - a.z = 0. \end{aligned}$$

Therefore $(Da)(G) \subset \ker T = S(E^*)$. Without less of generality we suppose that $S(E^*) = E^*$. Hence $D : \mathcal{A} \longrightarrow \mathcal{B}_{\mathfrak{A}}(G,E^*)$ is a bounded module derivation, and since $\mathcal{B}_{\mathfrak{A}}(G,E^*)$ is a dual commutative Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule, then there exists $Q_2 \in \mathcal{B}_{\mathfrak{A}}(G,E^*)$ with $Q_2(G) \subset \ker T$, and $Da = a.Q_2 - Q_2.a$. Set $Q = Q_1 - Q_2$, then $Q \in \mathcal{B}_{\mathfrak{A}}(G,F)$, and for each $a \in \mathcal{A}$ we have

$$\begin{aligned} a.Q &= a.Q_1 - a.Q_2 = a.Q_1 - Q_1.a + Q_1.a - a.Q_2 + Q_2.a - Q_2.a \\ &= Da + Q_1.a - Da - Q_2.a = Q.a. \qquad (2.10) \end{aligned}$$

Now, we should show that $Q \in {}_{\mathcal{A}}\mathcal{B}_{\mathfrak{A}}(G,F)$. By (2.10), for each $a \in \mathcal{A}$ and $x \in G$ we have

$$\langle x, a.Q\rangle = \langle x, Q.a\rangle = \langle a.x, Q\rangle.$$

Thus $Q \in {}_{\mathcal{A}}\mathcal{B}_{\mathfrak{A}}(G,F)$, and this means $\sum$ splits strongly. □

Let $\mathcal{A}$ be a Banach algebra, and $\mathcal{A}^{**}$ be the second dual space of $\mathcal{A}$. There are two products on $\mathcal{A}^{**}$; these products are denoted by $\square$ and $\diamond$, and are called the first and second *Arens* products (for more details see [5]).

Now, similar to amenable Banach algebras, we can prove the following Theorem for module amenability of Banach algebras:

Theorem 2.4. *Let $\mathcal{A}$ be a commutative Banach $\mathfrak{A}$-bimodule, and let I be a two-sided closed ideal in $\mathcal{A}$, which is a commutative Banach $\mathfrak{A}$-bimodule. If $\mathcal{A}$ is module amenable, then the following statements are equivalent:*

(i) I has a bounded approximate identity;

(ii) I is weakly complemented;

(iii) I is module amenable.

Proof. (i)⇒(iii). Let $\mathcal{X}$ be a I-essential module. By lemma 2.1 of [1], it suffices to show that $H^1_{\mathfrak{A}}(\mathcal{A},\mathcal{X}^*) = \{0\}$. Let $D : I \longrightarrow \mathcal{X}^*$ be a bounded module derivation. By (2.7) and (2.8), $\mathcal{X}$ is a unital Banach $\mathcal{I}$-bimodule, and by Theorem 2.1, there is a unique module derivation $\widetilde{D} : \mathcal{M}(I) \longrightarrow \mathcal{X}^*$ such that $\widetilde{D}|_I = D$. Then there is a regular continuous homomorphism $\theta : \mathcal{A} \longrightarrow \mathcal{I}$, and since $\mathcal{A}$ is module amenable, hence $\overline{\theta(\mathcal{A})}$ also is module amenable (Proposition 2.5 of [1]). Therefore $\widetilde{D}|_{\theta(\mathcal{A})}$ is inner. Thus D is an inner module derivation.

(iii)⇒(i). It is clear by Proposition 2.2 of [1].

(ii)⇒(i). Let I be a two-sided closed ideal in $\mathcal{A}$, then

$$\sum : 0 \longrightarrow I \xrightarrow{\imath} \mathcal{A} \xrightarrow{\pi} \mathcal{A}/I \longrightarrow 0$$

is a short exact sequence of commutative Banach $\mathcal{A}$-$\mathfrak{A}$-bimodules. Then dual sequence

$$\sum{}^* : 0 \longrightarrow (\mathcal{A}/I)^* \xrightarrow{\pi^*} \mathcal{A} \xrightarrow{\imath^*} I^* \longrightarrow 0$$

is a short exact sequence of commutative Banach $\mathcal{A}$-$\mathfrak{A}$-bimodules. Therefore $\sum^*$ is admissible (Theorem 2.8.31 of [4]), and since $\mathcal{A}$ is module amenable, then by

Proposition 2.2, $\sum^*$ splits strongly. Thus there exists $Q \in {}_{\mathcal{A}}\mathcal{B}_{\mathfrak{A}}(I^*, \mathcal{A}^*)$ such that $\imath^* \circ Q = id_{I^*}$. Also, since $\mathcal{A}$ is module amenable, then $\mathcal{A}$ has a left bounded approximate identity (Proposition 2.2 of [1]), and by Proposition 2.9.16 of [4], $(A^{**}, \diamond)$ has a left identity ($\diamond$ is the second Arens product on $\mathcal{A}^{**}$). Let e be the left identity of $(A^{**}, \diamond)$. Then

$$\begin{aligned} \langle a, \lambda \rangle &= \langle e.a, Q\lambda \rangle = \langle e, a.Q\lambda \rangle = \langle e, Q(a.\lambda) \rangle \\ &= \langle Q^*e, a.\lambda \rangle = \langle Q^*e.a, \lambda \rangle \qquad (a \in I, \lambda \in I^*). \end{aligned}$$

Then $a = Q^*e$, and this mean Q^*e is a left identity for $(I^{**}, \diamond)$. Hence I has a left bounded approximate identity (Proposition 2.9.16 of [4]). For right case, work is similar, therefore proof is complete.

(i)$\Rightarrow$(ii). It is clear by Theorem 2.9.58 of [4]. □

A semigroup S is an inverse semigroup if for each $s \in S$ there exists unique $s^* \in S$ with $ss^*s = s$, $s^*ss^* = s^*$. A convenient introduction to inverse semigroups may be found in [6]. The mapping $s \mapsto s^*$ is an involution on S, i.e. $s^{**} = s$ and $(st)^* = t^*s^*$ for all $s, t \in S$ (see [11]).

We denote by E_S the set of idempotents in S. Each idempotent of S is self-adjoint, and E_S is a commutative idempotent subsemigroup of S; in particular E_S is a semilattice. Now we are ready to give a new proof of Corollary 1.22 of [10]:

Theorem 2.5. *Let S be an amenable inverse semigroup, and let I be a two-sided ideal in S. Then I is amenable.*

Proof. Let S be inverse semigroup with the set of idempotents E_S. Consider $\ell^1(S)$ as a Banach module over $\ell^1(E_S)$ with the multiplication right action and the trivial left action. Then $\ell^1(S)$ is module amenable if and only if S is amenable (Theorem 3.1 of [1]). $\ell^1(I)$ is complemented in $\ell^1(S)$ as a Banach space, and is a commutative Banach $\ell^1(E_S)$-bimodule with the multiplication right action and the trivial left action. Then by Theorem 2.3, $\ell^1(I)$ is module amenable. Thus, by Theorem 3.1 of [1], I is amenable. □

3 Constant of Module Amenability

Let $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ be the projective module tensor product of $\mathcal{A}$ and $\mathcal{A}$. This is the quotient of the usual projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ by the closed ideal $\mathcal{I}$ generated by elements of the form $\alpha.a \otimes b - a \otimes b.\alpha$ for $\alpha \in \mathfrak{A}$, and $a, b \in \mathcal{A}$. We have

$(\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A})^* \cong \mathcal{L}_{\mathfrak{A}}(\mathcal{A},\mathcal{A}^*)$, where the right hand side is the space of all $\mathcal{A}$-module morphisms from $\mathcal{A}$ into $\mathcal{A}^*$ [12]. In particular $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule. Consider $\omega \in \mathcal{L}(\mathcal{A}\widehat{\otimes}\mathcal{A},\mathcal{A})$ defined by $\omega(a\otimes b) = ab$ for each $a, b \in \mathcal{A}$, and extended by linearity. Then both ω and its second conjugate $\omega^{**} \in \mathcal{L}((\mathcal{A}\widehat{\otimes}\mathcal{A})^{**},\mathcal{A}^{**})$ are $\mathfrak{A}$-module homomorphisms. Let $\mathcal{J}$ be the closed ideal of $\mathcal{A}$ generated by $\omega(\mathcal{I})$. We define $\widetilde{\omega} : \mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A} = \mathcal{A}\widehat{\otimes}\mathcal{A}/\mathcal{I} \longrightarrow \mathcal{A}/\mathcal{J}$ by

$$\widetilde{\omega}(a \otimes b + \mathcal{I}) = ab + \mathcal{J} \qquad\qquad (a, b \in \mathcal{A}).$$

The notion of C-amenability comes from [7] and the amenability constant $AM(\mathcal{A})$ was specifically introduced in [9]. The Banach algebra $\mathcal{A}$ is called C-amenable if has a bounded approximate diagonal (u_α) such that $\sup_\alpha \|u_\alpha\|_\pi \le C$ ($\|.\|_\pi$ is the projection norm). $AM(\mathcal{A})$ is the minimum of the appropriate constants C, and $AM(\mathcal{A}) \ge 1$. Accordingly to definition of C-amenability, we define C-module amenability of Banach algebras.

A bounded net $(\widetilde{u}_\alpha)$ in $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ is called a module approximate diagonal if $\widetilde{\omega}(\widetilde{u}_\alpha)$ is a bounded approximate identity of $\mathcal{A}/\mathcal{J}$ and

$$\lim_\alpha \|\widetilde{u}_\alpha.a - a.\widetilde{u}_\alpha\| = 0 \qquad (a \in \mathcal{A}).$$

Then we say $\mathcal{A}$ is C-module amenable if $\mathcal{A}$ has a module approximate diagonal (u_α) such that $\sup_\alpha \|u_\alpha\|_{\widetilde{\omega}} \le C$. Now we can consider the following Theorems.

Theorem 3.1. *Let $\mathcal{A}$ be module amenable Banach algebra with an identity $e_{\mathcal{A}}$, and $\mathcal{A}/\mathcal{J}$ has a bounded approximate identity. Then the following statements hold:*

(i) $AM(\mathcal{A}) \ge \|e_{\mathcal{A}}\|_{\mathcal{A}}$*;*

(ii) Let I be a two-sided closed with an identity e_I. Then I is module amenable with $AM(I) \le \|e_I\|AM(\mathcal{A})$.

Proof. (i) is clear. For (ii), by Theorem 2.4 module amenability of $\mathcal{A}$ implies module amenability of I. Since $\mathcal{A}$ is module amenable, then $\mathcal{A}$ has a bounded module approximate diagonal (Theorem 2.1 of [1]). Let (u_α) be a bounded approximate diagonal for $\mathcal{A}$ in $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ with $\sup_\alpha \|u_\alpha\| \le AM(\mathcal{A})$. For each α, set $v_\alpha = e_I.u_\alpha.e_I \in I\widehat{\otimes}_{\mathfrak{A}}I$. Where $I\widehat{\otimes}_{\mathfrak{A}}I$ is the projective module tensor product of I and I. This is the quotient of the usual projective tensor product $I\widehat{\otimes}I$ by closed ideal T generated by elements of the form $\beta.a \otimes b - a \otimes b.\beta$ for $\beta \in \mathfrak{A}$ and

$a, b \in I$. Similarly to $\mathcal{A}$, consider $\omega_I \in \mathcal{L}(I\widehat{\otimes}I, I)$. Let $\mathcal{T}$ be the closed ideal of I generated by $\omega(T)$. Define $\widetilde{\omega}_I : I\widehat{\otimes}_{\mathfrak{A}}I = I\widehat{\otimes}I/T \longrightarrow I/\mathcal{T}$ by

$$\widetilde{\omega}_I(a \otimes b + T) = ab + \mathcal{T} \qquad (a, b \in I).$$

Then we have

$$\widetilde{\omega}_I(v_\alpha) = \widetilde{\omega}_I(e_I.u_\alpha.e_I) \longrightarrow e_I + \mathcal{T}, \tag{3.1}$$

and

$$\begin{aligned}\lim_\alpha \|a.v_\alpha - v_\alpha.a\| &= \lim_\alpha \|a.e_I.u_\alpha.e_I - e_I.u_\alpha.e_I.a\| \\ &\leq \lim_\alpha \|e_I\|^2 \|a.u_\alpha - u_\alpha.a\| = 0. \end{aligned} \tag{3.2}$$

Also, we have

$$\begin{aligned}\lim_\alpha \|v_\alpha - e_I.u_\alpha\| &= \lim_\alpha \|e_I.u_\alpha.e_I - e_I.u_\alpha\| \\ &\leq \lim_\alpha \|e_I\|\|u_\alpha.e_I - e_I.u_\alpha\| = 0. \end{aligned} \tag{3.3}$$

Thus $\limsup_\alpha \|v_\alpha\| \leq \|e_I\| AM(\mathcal{A})$. By (2.11), (2.12), (2.13), and Theorem 2.1 of [1], proof is complete. □

Proposition 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-bimodules with compatible actions, and let $\mathcal{A}$ be C-module amenable. If there exists a continuous module homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ with dense range. Then $\mathcal{B}$ is $\|\varphi\|^2 C$-module amenable.*

Proof. Module amenability of $\mathcal{A}$ implies module amenability of $\mathcal{B}$ (Proposition 2.5 of [1]). Since $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a continuous Banach algebra homomorphism, then there exists a continuous module homomorphism $\varphi \otimes_{\mathfrak{A}} \varphi : \mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A} \longrightarrow \mathcal{B}\widehat{\otimes}_{\mathfrak{A}}\mathcal{B}$. Suppose that (u_α) is a module approximate diagonal for $\mathcal{A}$ such that $\sup_\alpha \|u_\alpha\| \leq C$. Set $(U_\alpha) = (\varphi \otimes_{\mathfrak{A}} \varphi)(u_\alpha)$. Then (U_α) is a module approximate diagonal for $\mathcal{B}$ and $\|U_\alpha\| \leq \|\varphi\|^2 C$. □

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