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Survey Article

Geometry of direct sums of Banach spaces

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Abstract: A brief survey is given of the geometry of various direct sums of finitely many Banach spaces with respect to a variety of generalizations of uniform convexity and uniform smoothness.

Keywords: Convexity, smoothness, direct sum

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1 Introduction

The notion of uniform convexity in Banach spaces is introduced by Clarkson in 1936. He also proved that the ℓ^p direct sum of finitely many uniformly convex Banach spaces is again uniformly convex. Since then the geometry of Banach spaces has been widely studied by many authors. Moreover, such results play an important role in the metric fixed point theory (for example see [12, 31]).

The purpose of this paper is to give a *brief survey* of the facts that have been discovered about the geometry of direct sums of Banach spaces. We are mainly interested in direct sums of finitely many Banach spaces rather than that of infinite

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ones. The restriction is imposed for reasons of time and the burden of technical details.

The paper is organized as follows: Section 2 contains the definitions of various convexity and smoothness considered in this survey. Section 3 deals with characterizations of geometric properties of direct sums of Banach spaces in terms of properties of each summand and of the substitution space.

We now recall some notion used throughout the paper. Let Z be a finite dimensional normed space $(\mathbb{R}^n, \|\cdot\|_Z)$, which has a *monotone norm*; that is,

$$\|(a_1, \ldots, a_n)\|_Z \leq \|(b_1, \ldots, b_n)\|_Z$$

if $0 \leq a_i \leq b_i$ for each $i = 1, \ldots, n$. We write $(X_1 \oplus \cdots \oplus X_n)_Z$ for the Z-direct sum of the Banach spaces $X_1, \ldots, X_n$ equipped with the norm

$$\|(x_1, \ldots, x_n)\| = \big\|\big(\|x_1\|_{X_1}, \ldots, \|x_n\|_{X_n}\big)\big\|_Z,$$

where $x_i \in X_i$ for each $i = 1, \ldots, n$.

One should notice that in defining $(X_1 \oplus \cdots \oplus X_n)_Z$, we only need to know the behavior of the Z-norm on $\mathbb{R}^n_+$. Consequently, in analyzing the extremal structure of the unit ball of $(X_1 \oplus \cdots \oplus X_n)_Z$, we can and do assume that the Z-norm is *absolute*; that is,

$$\|(a_1, \ldots, a_n)\|_Z = \big\|\big(|a_1|, \ldots, |a_n|\big)\big\|_Z \quad \text{for all } (a_1, \ldots, a_n) \in \mathbb{R}^n.$$

2 Definitions

Let X be a real Banach space with the unit sphere S_X and the closed unit ball B_X. The abbreviation of each definition will be used to denote that property. To gain a history of the births of the following geometric properties, the name of mathematician(s) who introduced each property and the corresponding paper are included after each definition.

(UC): X is *uniformly convex* if whenever $\{x_n\}$ and $\{y_n\}$ are sequences in S_X such that $\|x_n + y_n\| \to 2$, then $x_n - y_n \to 0$. [3, Clarkson]

(SC): X is *strictly convex* if whenever x and y are in S_X such that $\|x+y\| = 2$, then $x = y$.

(WUC): X is *weakly uniformly convex* if whenever $\{x_n\}$ and $\{y_n\}$ are sequences in S_X such that $\|x_n + y_n\| \to 2$, then $x_n - y_n \to 0$ weakly. [29, Šmul'yan]

(LUC): X is *locally uniformly convex* if whenever $\{x_n\}$ is a sequence in S_X and $x \in S_X$ such that $\|x_n + x\| \to 2$, then $x_n \to x$. [18, Lovaglia]

(WLUC): X is *weakly locally uniformly convex* if whenever $\{x_n\}$ is a sequence in S_X and $x \in S_X$ such that $\|x_n + x\| \to 2$, then $x_n \to x$ weakly. [17, Lindenstrauss]

(MLUC): X is *midpoint locally uniformly convex* if whenever $\{x_n\}$ and $\{y_n\}$ are sequences in B_X and $x \in S_X$ such that $\|2x - (x_n + y_n)\| \to 0$, then $x_n - y_n \to 0$. [1, Anderson]

(WMLUC): X is *weakly midpoint locally uniformly convex* if whenever $\{x_n\}$ and $\{y_n\}$ are sequences in B_X and $x \in S_X$ such that $\|2x - (x_n + y_n)\| \to 0$, then $x_n - y_n \to 0$ weakly. [27, Smith]

(UCED): X is *uniformly convex in every direction* if whenever nonzero z in X and $\{x_n\}$ and $\{y_n\}$ are sequences in B_X such that $\|x_n + y_n\| \to 2$ and $x_n - y_n = \alpha_n z$, then $\alpha_n \to 0$. [11, Garkavi]

(Sch): X has the *Schur property* if whenever $x \in X$ and $\{x_n\}$ is a sequence in X such that $x_n \to x$ weakly, then $x_n \to x$.

(KK): X has the *Kadec–Klee property* if whenever $x \in X$ and $\{x_n\}$ is a sequence in X such that $x_n \to x$ weakly and $\|x_n\| \to \|x\|$, then $x_n \to x$.

(UKK): X is *uniform Kadec–Klee property* if for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $x \in X$ and $\{x_n\}$ is a sequence in X such that $x_n \to x$ weakly and $\|x_n - x_m\| \geq \varepsilon$ for all $m \neq n$, then $\|x\| \leq 1 - \delta$. [13, Huff]

(NUC): X has the *nearly uniformly convex* if for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $x \in X$ and $\{x_n\}$ is a sequence in B_X such that $\|x_n - x_m\| \geq \varepsilon$ for all $m \neq n$, then $\operatorname{conv}\{x_n\} \cap (1-\delta)B_X \neq \varnothing$. [13, Huff]

(D): X has the *drop property* if whenever a closed subset C of X is disjoint from B_X, then there is x such that the drop $D(x, C) := \operatorname{conv}(\{x\} \cup B_X)$ intersects C only at x. [4, Daneš]

(U): X is a *U-space* if for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever x and y are elements in S_X such that $\|x+y\| \geq 2(1-\delta)$, then there is a supporting functional f at x such that $f(y) \geq 1 - \varepsilon$. [16, Lau]

(u): X is a *u-space* if whenever x and y are in B_X such that $\|x + y\| = 2$, then the sets of supporting functionals at x and at y coincide. [6, Dhompongsa, Kaewkhao and Saejung]

(US): X is *uniformly smooth* if the limit $\lim_{t\to 0} \frac{1}{t}(\|x+ty\| + \|x-ty\| - 2) = 0$ for all $x, y \in S_X$ and the convergence is uniform for $x, y \in S_X$. [30, Šmul'yan]

(S): X is *smooth* if there is only one supporting functional at every $x \in S_X$.

Relations between properties above are shown in the following implication diagrams.

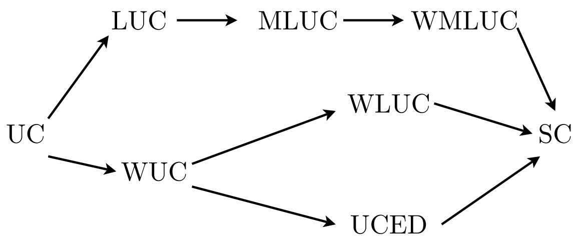

Figure 1:

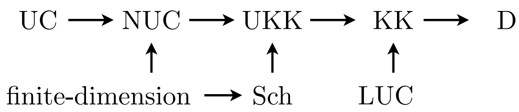

Figure 2:

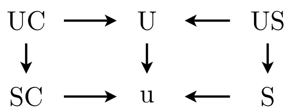

Figure 3:

3 Convexity and smoothness

Let $X_1, \ldots, X_n$ be Banach spaces. Then we have the following results:

$(X_1 \oplus \cdots \oplus X_n)_Z$	Each X_j	Z	Related reference(s)
UC	UC	SC	[5, 7, 14, 25, 26]
SC	SC	SC	[14, 32]
WUC	WUC	SC	[9]
LUC	LUC	SC	[9, 18]
WLUC	WLUC	SC	[9]
MLUC	MLUC	SC	[9]
WMLUC	WMLUC	SC	[9]
UCED	UCED	SC	[7, 28]
U	U	u	[6, 19]
u	u	u	[6, 19]
US	US	S	[6, 14]
S	S	S	[6, 20]

From the table above, for example, it reads $(X_1 \oplus \cdots \oplus X_n)_Z$ is UC if and only if each X_j is UC and Z is SC.

Before moving to the next result, we introduce the following property of Z.

(j-th SM): Z is *strictly monotone in the j-th coordinate* if whenever $0 \le a_i \le b_i$ for each $i = 1, \ldots, n$ and $0 \le a_j < b_j$, then $\|(a_1, \ldots, a_n)\|_Z < \|(b_1, \ldots, b_n)\|_Z$.

$(X_1 \oplus \cdots \oplus X_n)_Z$	Each X_j	For each j, Z is j-th SM or X_j	Related reference(s)
KK	KK	Sch	[8]
UKK	UKK	Sch	[8, 15]
NUC	NUC	Finite dimensional	[8, 15]
D	D	Finite dimensional	[8]

From the table above, for example, it reads $(X_1 \oplus \cdots \oplus X_n)_Z$ has KK if and only if each X_j has KK and for each j, Z is j-th SM or X_j has Sch.

Finally, we consider another geometric property which is related to fixed point property. Let us recall the following definition. For $x^*, y^* \in S_{X^*}$ and $0 \le \delta \le 1$, define

$$S(x^*, y^*, \delta) = \{x \in B_X : \min\{x^*(x), y^*(x)\} \ge 1 - \delta\}.$$

A Banach space X is said to have a *crease* if there are $x^*, y^* \in S_{X^*}$ such that $x^* \ne y^*$ and $\mathrm{diam} S(x^*, y^*, 0) > 0$. X is said to be *noncreasy* (Nc) if it has no creases. A uniform version of noncreasy is the following: X is *uniformly noncreasy* (UNc) if for each $\varepsilon > 0$, there is a $\delta > 0$, so that $\operatorname{diam} S(x^*, y^*, \delta) \le \varepsilon$ whenever $x^*, y^* \in S_{X^*}$ with $\|x^* - y^*\| \ge \varepsilon$. The reader is directed to the paper of Prus [23] where the basic properties of Nc and UNc spaces are derived. The following are proved in [9].

Theorem 3.1. *Let $X_1, \ldots, X_n$ be Banach spaces. Then $(X_1 \oplus \cdots \oplus X_n)_Z$ is Nc if and only if all the following conditions are satisfied:*

(a) *all the X_i and Z are Nc;*

(b) *if there exists $1 \le i_0 \le n$ such that X_{i_0} is not SC, then all the X_i, with $i \ne i_0$, are S, and each point $(a_1, \ldots, a_n) \in S_Z$ with $a_{i_0} \ne 0$, is a smooth point of B_Z;*

(c) *if there exists $1 \le i_0 \le n$ such that X_{i_0} is not S, then all the X_i, with $i \ne i_0$, are SC, and each point $(a_1, \ldots, a_n) \in S_Z$ with $a_{i_0} \ne 0$, is an extreme point of B_Z.*

By using ultrapowers as in [24], we obtain a uniform version of Theorem 3.1.

Theorem 3.2. *Let $X_1, \ldots, X_n$ be Banach spaces. Then $(X_1 \oplus \cdots \oplus X_n)_Z$ is UNc if and only if all the following conditions are satisfied:*

(a') *all the X_i and Z are UNc;*

(b') *if there exists $1 \le i_0 \le n$ such that X_{i_0} is not UC, then all the X_i, with $i \ne i_0$, are US, and each point $(a_1, \ldots, a_n) \in S_Z$ with $a_{i_0} \ne 0$, is a smooth point of B_Z;*

(c') *if there exists $1 \le i_0 \le n$ such that X_{i_0} is not US, then all the X_i, with $i \ne i_0$, are UC, and each point $(a_1, \ldots, a_n) \in S_Z$ with $a_{i_0} \ne 0$, is an extreme point of B_Z.*

We now recall some geometric properties of Banach spaces which are mentioned above. Let X be a Banach space and $x \in S_X$.

(a) x is called an *extreme point* of B_X if whenever y and z are elements of B_X with $x = \frac{y+z}{2}$, then $y = z = x$.

(b) x is called a *smooth point* of B_X if there is a unique supporting functional at x.

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